

15.433 INVESTMENTS
Class 4: Portfolio Theory
Part 2: Extensions

Spring 2003



Introduction

- Should someone with longer investment horizon put more in the stock market?
- Is there value in dynamic re-balancing?
- How do market crashes affect investment behavior above and beyond their impact on mean and variance?

One Day in LTCM

On Aug. 17, 1998, Russia defaulted on its local debt. Friday August 21 at LTCM, Greenwich, CT:

"When he saw the quotation for U.S. swap spreads, he stared at his screen in disbelief. On an active day, U. S. ' swap spreads might change by as much as a point. But on this morning, swap spreads were wildly oscillating over a range of 20 points. "That Friday, Long? Term lost money wherever it looked. Credit spreads simply exploded. ? Though these moves may seem small in absolute terms, the effect on Long? Term was magnified by the fund's potent level of leverage and its immense position size." "Long-Term, which had calculated with such mathematical certainty that it was unlikely to lose more than \$35 million on any single day, had just dropped \$553 million on that one Friday in August."

"When Genius Failed" by Roger Lowenstein

Fat Tails

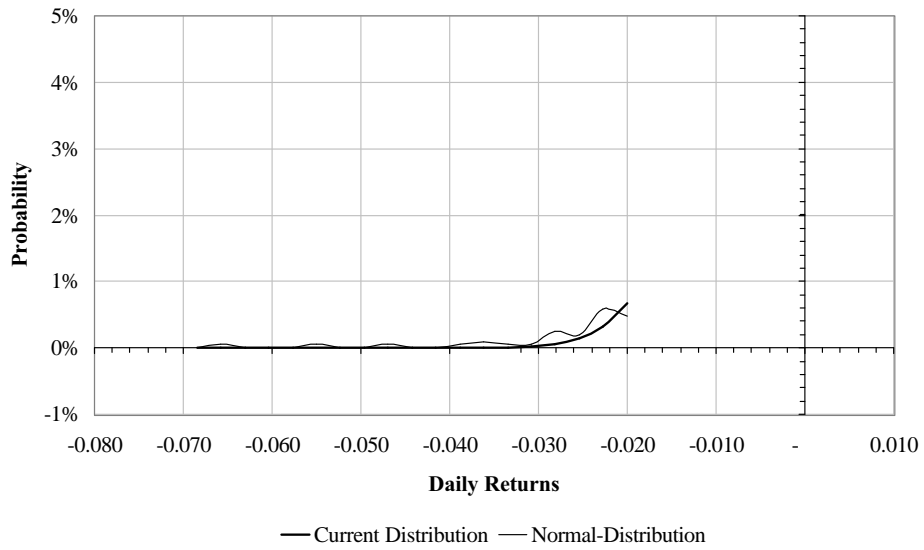


Figure 1: Left fat tail of daily return distribution of S&P 500 index.

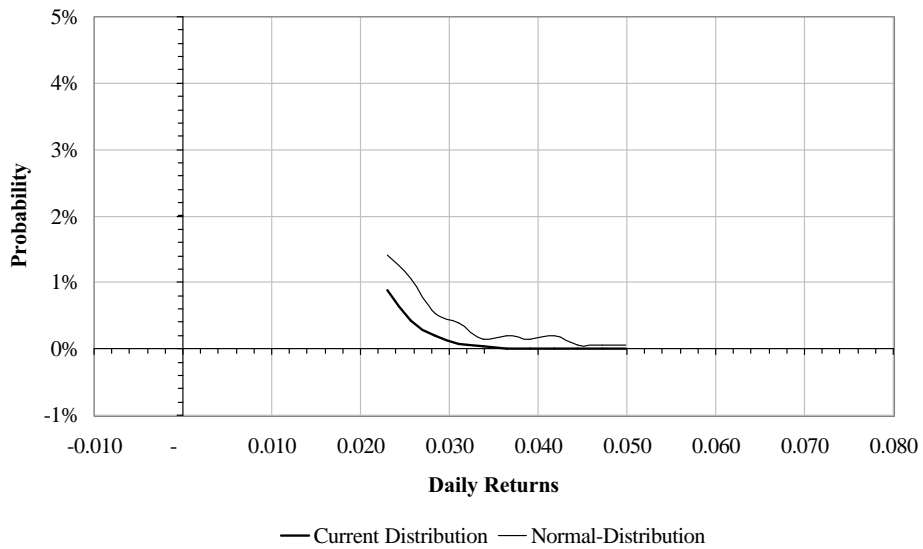


Figure 2: Right fat tail of daily return distribution of S&P 500 index.

Quantify Fat Tails Skewness

Skewness

$$skew(x) = \frac{E(x - E(x))^3}{std(x)^3} \quad (1)$$

Kurtosis

$$kurto(x) = \frac{E(x - E(x))^4}{std(x)^4} \quad (2)$$

What is the skewness of a standard normal?

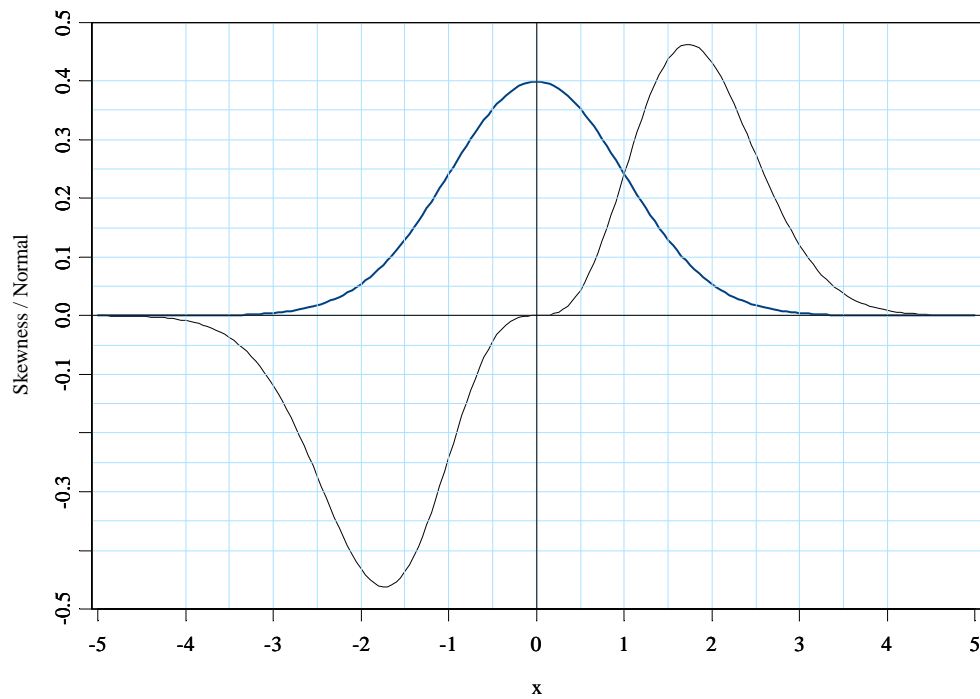


Figure 3: Skewness of a standard normal distribution, *splus*.

What is the kurtosis of standard normal?

Caution: skewness and kurtosis might not exist for some random variables.

A Model with Event Risk

Daily Return r :

$$r = x + y$$

x is the "normal" component:

y is the "jump" component:

$$y = \begin{cases} J & \text{with probability } p \\ 0 & \text{with probability } 1-p \end{cases}$$

x and y are independent.

Moments of the Crash Model

mean

$$E(r) = \mu + J \cdot p \quad (3)$$

variance:

$$\text{var}(r) = \sigma^2 + J^2 \cdot p \cdot (1 - p) \quad (4)$$

skewness:

$$\text{skew}(r) = \frac{J^3 \cdot p \cdot (1 - p) \cdot (1 - 2p)}{\sqrt[3]{\text{var}(r^2)}} \quad (5)$$

The Skewness of the Crash Model

Model	P	J	skew
once a month	1/12	-2%	-1.0
once a year	1/365	-10%	-5.6
once a lifetime	1/365/100	-50%	-7.1

Each jump model is calibrated so that the annualized expected return is 12%, and the annualized volatility is 15%.

That is, the returns from the three models are equally attractive to a mean-variance investor!

Skewness Preference

A utility function that captures the risk attitude of an investor in three ways:

$$U(r) = E(r) - \frac{1}{2} \cdot A \cdot \text{var}(r) + \frac{1}{6} \cdot B \cdot E(r - E(r))^3 \quad (6)$$

1. the expected return is desirable;
2. the variance of return is undesirable ($A > 0$);
3. positive skewness is desirable, negative skewness is not ($B > 0$).

If we set $B = 0$, we are back to a mean-variance investor.

Modifying the Original Problem

Two modifications of our original problem:

1. Investor has a preference for skewness;

$$U(r) = E(r) - \frac{1}{2} \cdot A \cdot \text{var}(r) + \frac{1}{6} \cdot B \cdot E(r - E(r))^3 \quad (7)$$

2. The risky asset r_P is negatively skewed:

$$\text{skew} = -2$$

The optimization problem:

$$\max_{y \in \mathbb{R}} U(r_y) \quad (8)$$

Just as before, r_P is the portfolio return:

$$r_y = (1 - y) \cdot r_f + y \cdot r_P \quad (9)$$

Formalizing the Problem

As before, we calculate the utility $U(r_y)$ of an investor who invests a fraction y of his wealth in the risky asset r_p , leaving the rest in the riskfree account r_f :

$$r_y = r_f + y \cdot (r_p - r_f) \quad (10)$$

Two old pieces of $U(r_y)$:

$$E(r_y) = 0.07 + 0.08 \cdot y; \quad \text{var}(r_y) = 0.22^2 \cdot y^2 \quad (11)$$

The new piece of $U(r_y)$:

$$E(r_y - E(r_y))^3 = y^3 \cdot E_p - E((r_p)^3) \quad (12)$$

Recall that by definition

$$E(x - E(x))^3 = \text{skew}(x) \cdot \text{std}(x)^3 \quad (13)$$

we therefore have:

$$E(r_y - E(r_y))^3 = 0.22^3 \cdot (-2) \cdot (y)^3 \quad (14)$$

The Optimization Machine

Armed with all three pieces of $U(r_y)$, we write our optimization problem in mathematical terms:

$$\begin{aligned} \max_{y \in \mathbb{R}} \quad & U(r_y) \end{aligned} \tag{15}$$

$$f(y) = 0.07 + 0.08 \cdot y - \frac{1}{2} \cdot A \cdot 0.22^2 + \frac{1}{6} \cdot B \cdot 0.22^3 \cdot (-2) \cdot y^3 \tag{16}$$

In our original problem, $f(y)$ is a linear-quadratic function of y .

What about now?

Optimization: Analytical Method

The mathematical foundation:

- let y^* be the solution of $f'(y) = 0$
- let $f''(y^*) < 0$, then y^* is truly the optimal solution.

Let's take some derivatives:

$$f'(y) = 0.08 - 0.22^2 \cdot A \cdot y + \frac{1}{2} \cdot 0.22^3 \cdot (-2) \cdot B \cdot y^2 \quad (17)$$

$$f''(y) = -0.22^2 \cdot A + \frac{1}{2} \cdot 0.22^3 \cdot (-2) \cdot B \cdot y \quad (18)$$

What do you think? How do we look for the optimal y^* ? Does it even exist?

Optimization: A Pictorial Solution

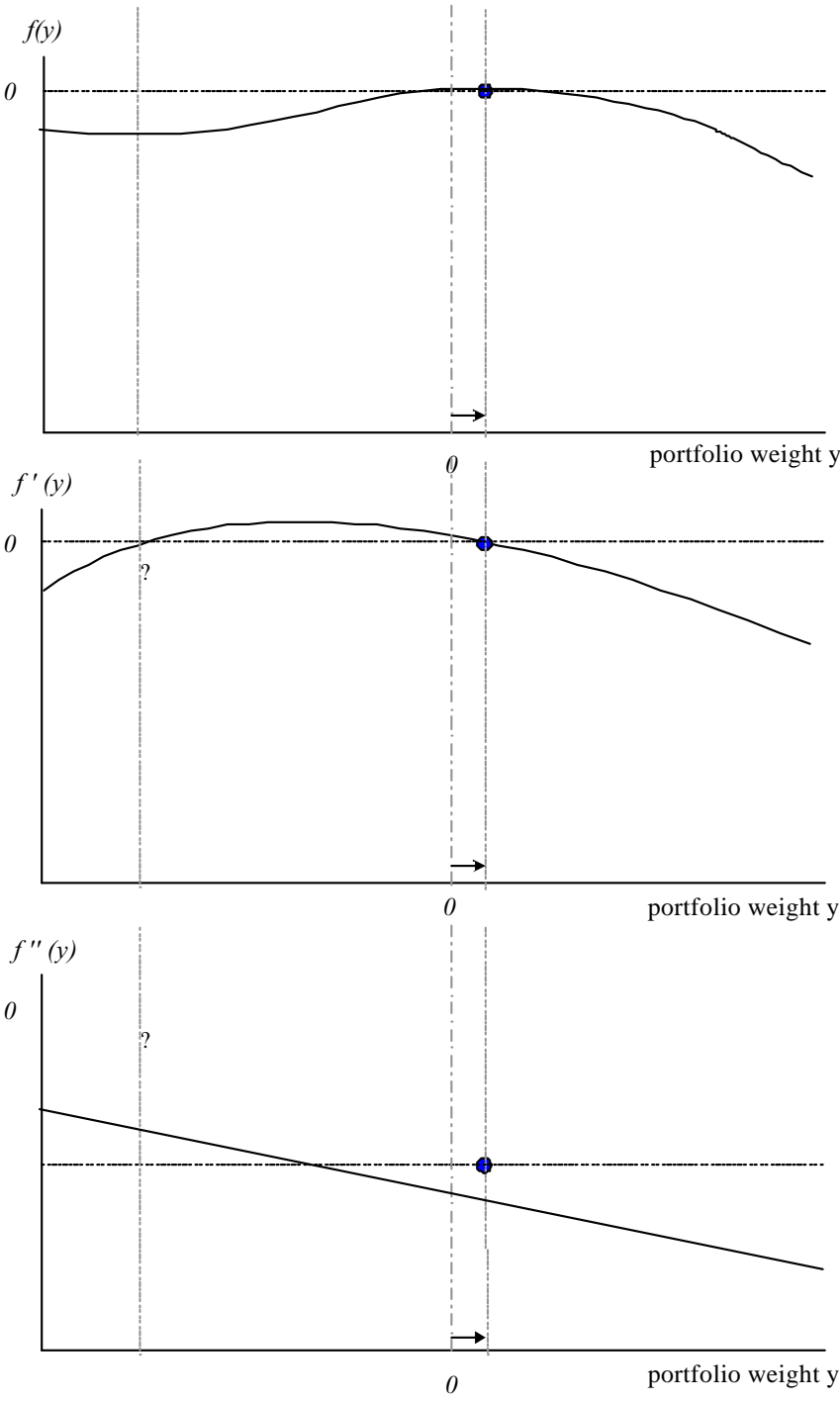


Figure 4: Risk aversion and optimal weight.

The variance aversion coefficient is set at $A = 4$, and the skew preference is set at $B = 5$.

The optimal portfolio weight is $y^* = 0.37$.

Recall that, given the same choice set (r_p and r_f), a mean-variance investor with $A = 4$ will invest $y^* = 0.41$.

Determinants of Optimal Allocation

What we already know:

1. a more variance-averse investor tends to invest less in the risky asset: $A \uparrow \Rightarrow y^* \downarrow$
2. given a higher risk premium, a risk-averse investor will invest more in the risky asset: $(E(r_p) - r_f) \uparrow \Rightarrow y^* \uparrow$
3. given the same risk premium, if the risky asset becomes more volatile, a risk-averse investor will reduce his holdings in it: $std(r_p) \uparrow \Rightarrow y^* \downarrow$

New Results

An investor who is more averse to negative skewness tends to invest less in a negatively skewed risky asset: $B \uparrow \Rightarrow y^* \downarrow$

Given the same risk premium and volatility, if a risky asset becomes more negatively skewed, an investor with skewness preference will reduce his holdings: $skew(r_p) \downarrow \Rightarrow y^* \downarrow$

The Horizon Effect

To study the horizon effect, we need to understand how stock returns aggregate over time.

Let's start with the simplest model.

$$r_1 = \mu + \sigma \cdot \varepsilon_1$$

$$r_2 = \mu + \sigma \cdot \varepsilon_2$$

$$r_3 = \mu + \sigma \cdot \varepsilon_3$$

...

The daily shocks ε_t are independent and standard normally distributed.

Focus:

BKM Appendix A, B of Chapter

- 6 p. 172 to 177 middle (probability, distribution, skewness [no equation], normal distribution)
- p. 178 middle to 181 middle (utility, utility function)

Reader: Black (1995)

type of potential questions: -

Questions for Next

Please read:

- BKM Chapter 8,
- Kritzman (1994), and
- Kritzman (1991)

Think about the following questions:

- What do we mean by diversification?
- When does diversification work, when doesn't it?
- Can you think of other real life examples (non?financial) when the principle of diversification is used?