Stochastic Calculus and Option Pricing

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15.450, Fall 2010
Outline

1. Stochastic Integral
2. Itô’s Lemma
3. Black-Scholes Model
4. Multivariate Itô Processes
5. SDEs
6. SDEs and PDEs
7. Risk-Neutral Probability
8. Risk-Neutral Pricing
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Consider a random walk $x_{t+n\Delta t}$ with equally likely increments of $\pm \sqrt{\Delta t}$.

Let the time step of the random walk shrink to zero: $\Delta t \to 0$.

The limit is a continuous-time process called *Brownian motion*, which we denote $Z_t$, or $Z(t)$.

We always set $Z_0 = 0$.

Brownian motion is a basic building block of continuous-time models.
Properties of Brownian Motion

- Brownian motion has independent increments: if \( t < t' < t'' \), then \( Z_{t'} - Z_t \) is independent of \( Z_{t''} - Z_{t'} \).
- Increments of the Brownian motion have normal distribution with zero mean, and variance equal to the time interval between the observation points

\[
Z_{t'} - Z_t \sim \mathcal{N}(0, t' - t)
\]

Thus, for example,

\[
E_t[Z_{t'} - Z_t] = 0
\]

Intuition: Central Limit Theorem applied to the random walk.

- Trajectories of the Brownian motion are continuous.
- Trajectories of the Brownian motion are **nowhere differentiable**, therefore standard calculus rules do not apply.
Ito Integral

- Ito integral, also called the stochastic integral (with respect to the Brownian motion) is an object

\[ \int_0^t \sigma_u \, dZ_u \]

where \( \sigma_u \) is a stochastic process.

- Important: \( \sigma_u \) can depend on the past history of \( Z_u \), but it cannot depend on the future. \( \sigma_u \) is called adapted to the history of the Brownian motion.

- Consider discrete-time approximations

\[
\sum_{i=1}^{N} \sigma_{(i-1)\Delta t}(Z_{i\Delta t} - Z_{(i-1)\Delta t}), \quad \Delta t = \frac{t}{N}
\]

and then take the limit of \( N \to \infty \) (the limit must be taken in the mean-squared-error sense).

- The limit is well defined, and is called the Ito integral.
Properties of Itô Integral

- Itô integral is linear

\[ \int_0^t (a_u + c \times b_u) \, dZ_u = \int_0^t a_u \, dZ_u + c \int_0^t b_u \, dZ_u \]

- \( X_t = \int_0^t \sigma_u \, dZ_u \), is continuous as a function of time.

- Increments of \( X_t \) have conditional mean of zero (under some technical restrictions on \( \sigma_u \)):

\[ E_t[X_{t'} - X_t] = 0, \quad t' > t \]

Note: a sufficient condition for \( E_t \left[ \int_t^T \sigma_u \, dZ_u \right] = 0 \) is \( E_0 \left[ \int_0^T \sigma_u^2 \, du \right] < \infty \).

- Increments of \( X_t \) are uncorrelated over time.

- If \( \sigma_t \) is a deterministic function of time and \( \int_0^t \sigma_u^2 \, du < \infty \), then \( X_t \) is normally distributed with mean zero, and variance

\[ E_0[X_t^2] = \int_0^t \sigma_u^2 \, du \]
Itô Processes

- An Itô process is a continuous-time stochastic process $X_t$, or $X(t)$, of the form

$$\int_0^t \mu_u \, du + \int_0^t \sigma_u \, dZ_u$$

- $\mu_u$ is called the instantaneous drift of $X_t$ at time $u$, and $\sigma_u$ is called the instantaneous volatility, or the diffusion coefficient.

- $\mu_t \, dt$ captures the expected change of $X_t$ between $t$ and $t + dt$.

- $\sigma_t \, dZ_t$ captures the unexpected (stochastic) component of the change of $X_t$ between $t$ and $t + dt$.

- Conditional mean and variance:

$$E_t(X_{t+dt} - X_t) = \mu_t \, dt + o(dt), \quad E_t[(X_{t+dt} - X_t)^2] = \sigma_t^2 \, dt + o(dt)$$
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Quadratic Variation

- Consider a time discretization

\[ 0 = t_1 < t_2 < \ldots < t_N = T, \quad \max_{n=1,\ldots,N-1} |t_{n+1} - t_n| < \Delta \]

- Quadratic variation of an Itô process \( X(t) \) between 0 and \( T \) is defined as

\[ [X]_T = \lim_{\Delta \to 0} \sum_{n=1}^{N-1} |X(t_{n+1}) - X(t_n)|^2 \]

- For the Brownian motion, quadratic variation is deterministic:

\[ [Z]_T = T \]

To see the intuition, consider the random-walk approximation to the Brownian motion: each increment equals \( \sqrt{t_{n+1} - t_n} \) in absolute value.
Quadratic Variation

- Quadratic variation of an Itô process

\[ X_t = \int_0^t \mu_u \, du + \int_0^t \sigma_u \, dZ_u \]

is given by

\[ [X]_T = \int_0^T \sigma_t^2 \, dt \]

- Heuristically, the quadratic variation formula states that

\[ (dZ_t)^2 = dt, \quad dt \, dZ_t = o(dt), \quad (dt)^2 = o(dt) \]

- Random walk intuition:

\[ |dZ_t| = \sqrt{dt}, \quad |dt \, dZ_t| = (dt)^{3/2} = o(dt), \quad dZ_t^2 = dt \]

- Conditional variance of the Itô process can be estimated by approximating its quadratic variation with a discrete sum. This is the basis for variance estimation using high-frequency data.
Itô’s Lemma

- Itô’s Lemma states that if $X_t$ is an Itô process,

$$X_t = \int_0^t \mu_u \, du + \int_0^t \sigma_u \, dZ_u$$

then so is $f(t, X_t)$, where $f$ is a sufficiently smooth function, and

$$df(t, X_t) = \left( \frac{\partial f(t, X_t)}{\partial t} + \frac{\partial f(t, X_t)}{\partial X_t} \mu_t + \frac{1}{2} \frac{\partial^2 f(t, X_t)}{\partial X_t^2} \sigma^2_t \right) dt + \frac{\partial f(t, X_t)}{\partial X_t} \sigma_t \, dZ_t$$

- Itô’s Lemma is, heuristically, a second-order Taylor expansion in $t$ and $X_t$, using the rule that

$$(dZ_t)^2 = dt, \quad dt \, dZ_t = o(dt), \quad (dt)^2 = o(dt)$$
Itô’s Lemma

Using the Taylor expansion,

\[
df(t, X_t) \approx \frac{\partial f(t, X_t)}{\partial t} \, dt + \frac{\partial f(t, X_t)}{\partial X_t} \, dX_t + \frac{1}{2} \frac{\partial^2 f(t, X_t)}{\partial t^2} \, dt^2 + \frac{1}{2} \frac{\partial^2 f(t, X_t)}{\partial X_t^2} (dX_t)^2 + \frac{\partial^2 f(t, X_t)}{\partial t \partial X_t} \, dt \, dX_t
\]

\[
= \frac{\partial f(t, X_t)}{\partial t} \, dt + \frac{\partial f(t, X_t)}{\partial X_t} \, \mu_t \, dt + \frac{\partial f(t, X_t)}{\partial X_t} \, \sigma_t \, dZ_t + \frac{1}{2} \frac{\partial^2 f(t, X_t)}{\partial X_t^2} \, \sigma_t^2 \, dt + o(dt)
\]

Short-hand notation

\[
df(t, X_t) = \frac{\partial f(t, X_t)}{\partial t} \, dt + \frac{\partial f(t, X_t)}{\partial X_t} \, dX_t + \frac{1}{2} \frac{\partial^2 f(t, X_t)}{\partial X_t^2} (dX_t)^2
\]
Itô's Lemma

Example

- Let $X_t = \exp(at + bZ_t)$.
- We can write $X_t = f(t, Z_t)$, where

$$f(t, Z_t) = \exp(at + bZ_t)$$

- Using

$$\frac{\partial f(t, Z_t)}{\partial t} = af(t, Z_t), \quad \frac{\partial f(t, Z_t)}{\partial Z_t} = bf(t, Z_t), \quad \frac{\partial^2 f(t, Z_t)}{\partial Z_t^2} = b^2 f(t, Z_t)$$

Itô's Lemma implies

$$dX_t = \left( a + \frac{b^2}{2} \right) X_t \, dt + bX_t \, dZ_t$$

- Expected growth rate of $X_t$ is $a + b^2 / 2$. 
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The Black-Scholes Model of the Market

- Consider the market with a constant risk-free interest rate \( r \) and a single risky asset, the stock.
- Assume the stock does not pay dividends and the price process of the stock is given by
  \[
  S_t = S_0 \exp \left( \left( \mu - \frac{1}{2} \sigma^2 \right) t + \sigma Z_t \right)
  \]
- Because Brownian motion is normally distributed, using
  \[E_0[\exp(Z_t)] = \exp(t/2),\]
  find
  \[E_0[S_t] = S_0 \exp(\mu t)\]
- Using Itô’s Lemma (check)
  \[
  \frac{dS_t}{S_t} = \mu \, dt + \sigma \, dZ_t
  \]
- \( \mu \) is the expected continuously compounded stock return, \( \sigma \) is the volatility of stock returns.
- Stock returns have constant volatility.
Dynamic Trading

- Consider a trading strategy with continuous rebalancing.
- At each point in time, hold $\theta_t$ shares of stocks in the portfolio.
- Let the portfolio value be $W_t$. Then $W_t - \theta_t S_t$ dollars are invested in the short-term risk-free bond.
- Portfolio is self-financing: no exogenous incoming or outgoing cash flows.
- Portfolio value changes according to

\[ dW_t = \theta_t dS_t + (W_t - \theta_t S_t) r \, dt \]

- Discrete-time analogy

\[ W_{t+\Delta t} - W_t = \theta_t (S_{t+\Delta t} - S_t) + (W_t - \theta_t S_t) \left( \frac{B_{t+\Delta t}}{B_t} - 1 \right) \]
Option Replication

- Consider a European option with the payoff $H(S_T)$.
- We will construct a self-financing portfolio replicating the payoff of the option.
- Look for the portfolio such that
  \[ W_t = f(t, S_t) \]
  for some function $f(t, S_t)$.
- By Law of One Price, $f(t, S_t)$ must be the price of the option at time $t$, being the cost of a trading strategy with an identical payoff.
- Note that the self-financing condition is important for the above argument: we do not want the portfolio to produce intermediate cash flows.
Option Replication

Apply Itô’s Lemma to portfolio value, \( W_t = f(t, S_t) \):

\[
dW_t = \theta_t dS_t + (W_t - \theta_t S_t) r \, dt = \frac{\partial f}{\partial t} \, dt + \frac{\partial f}{\partial S_t} \, dS_t + \frac{1}{2} \frac{\partial^2 f}{\partial S_t^2} \, (dS_t)^2
\]

where \((dS_t)^2 = \sigma^2 S_t^2 \, dt\)

The above equality holds at all times if

\[
\theta_t = \frac{\partial f(t, S_t)}{\partial S_t}, \quad \frac{\partial f(t, S_t)}{\partial t} + \frac{1}{2} \frac{\partial^2 f(t, S_t)}{\partial S_t^2} \sigma^2 S_t^2 - r \left( f(t, S_t) - \frac{\partial f}{\partial S_t} S_t \right) = 0
\]

If we can find the solution \( f(t, S) \) to the PDE

\[
-r f(t, S) + \frac{\partial f(t, S)}{\partial t} + rS \frac{\partial f(t, S)}{\partial S} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 f(t, S)}{\partial S^2} = 0
\]

with the boundary condition \( f(T, S) = H(S) \), then the portfolio with

\[
W_0 = f(0, S_0), \quad \theta_t = \frac{\partial f(t, S_t)}{\partial S_t}
\]

replicates the option!
Black-Scholes Option Price

We conclude that the option price can be computed as a solution of the Black-Scholes PDE

\[-rf + \frac{\partial f}{\partial t} + rS \frac{\partial f}{\partial S} + \frac{1}{2} \frac{\partial^2 f}{\partial S^2} \sigma^2 S^2 = 0\]

If the option is a European call with strike \(K\), the PDE can be solved in closed form, yielding the Black-Scholes formula:

\[C(t, S_t) = S_t N(z_1) - \exp(-r(T-t))K N(z_2),\]

where \(N(.)\) is the cumulative distribution function of the standard normal distribution,

\[z_1 = \log \left( \frac{S_t}{K} \right) + \left( r + \frac{1}{2} \sigma^2 \right) (T - t) \frac{\sigma \sqrt{T - t}}{\sigma \sqrt{T - t}},\]

and

\[z_2 = z_1 - \sigma \sqrt{T - t}.

Note that \(\mu\) does not enter the PDE or the B-S formula. This is intuitive from the perspective of risk-neutral pricing. Discuss later.
The replicating strategy requires holding

\[ \theta_t = \frac{\partial f(t, S_t)}{\partial S_t} \]

stock shares in the portfolio. \( \theta_t \) is called the option’s delta.

It is possible to replicate any option in the Black-Scholes setting because

- Price of the stock \( S_t \) is driven by a Brownian motion;
- Rebalancing of the replicating portfolio is continuous;
- There is a single Brownian motion affecting the payoff of the option and the price of the stock. More on this later, when we cover multivariate Itô processes.
Consider a model of the term structure of default-free bond yields.

Assume that the short-term interest rate follows

\[ dr_t = \alpha(r_t) \, dt + \beta(r_t) \, dZ_t \]

Let \( P(t, \tau) \) denote the time-\( t \) price of a discount bond with unit face value maturing at time \( \tau \).

We want to construct an arbitrage-free model capturing, simultaneously, the dynamics of bond prices of many different maturities.
Single-Factor Term Structure Model

- Assume that bond prices can be expressed as a function of the short rate only (single-factor structure)

\[ P(t, \tau) = f(t, r_t, \tau) \]

- Itô formula implies that

\[
dP(t, \tau) = \left( \frac{\partial f}{\partial t} + \frac{\partial f}{\partial r} \alpha(r_t) + \frac{1}{2} \frac{\partial^2 f}{\partial r^2} \beta^2(r_t) \right) dt + \frac{\partial f}{\partial r} \beta(r_t) dZ_t
\]

- Consider a self-financing portfolio which invests \( P(t, \tau) / \sigma_t^\tau \) dollars in the bond maturing at \( \tau \), and \(-P(t, \tau') / \sigma_t'^\tau \) dollars in the bond maturing at \( \tau' \).

- Self-financing requires that the investment in the risk-free short-term bond is

\[
W_t = \frac{P(t, \tau)}{\sigma_t^\tau} + \frac{P(t, \tau')}{\sigma_t'^\tau}
\]
Single-Factor Term Structure Model

- Portfolio value evolves according to

\[
dW_t = \left( W_t - \frac{P(t, \tau)}{\sigma_t^\tau} + \frac{P(t, \tau')}{\sigma_t'^\tau} \right) rt + \frac{\mu_t^\tau}{\sigma_t^\tau} - \frac{\mu_t'^\tau}{\sigma_t'^\tau} \right) dt + \left[ \frac{\sigma_t^\tau}{\sigma_t'^\tau} - \frac{\sigma_t'^\tau}{\sigma_t'^\tau} \right] dZ_t
\]

- Portfolio value changes are instantaneously risk-free.
- To avoid arbitrage, the portfolio value must grow at the risk-free rate:

\[
\left( W_t - \frac{P(t, \tau)}{\sigma_t^\tau} + \frac{P(t, \tau')}{\sigma_t'^\tau} \right) rt + \frac{\mu_t^\tau}{\sigma_t^\tau} - \frac{\mu_t'^\tau}{\sigma_t'^\tau} = W_t r_t
\]
**Single-Factor Term Structure Model**

- We conclude that
  \[
  \frac{\mu_t^\tau - r_t P(t, \tau)}{\sigma_t^\tau} = \frac{\mu_t^{\tau'} - r_t P(t, \tau')}{\sigma_t^{\tau'}}, \quad \text{for any } \tau \text{ and } \tau'
  \]

- Assume that, for some \( \tau' \),
  \[
  \frac{\mu_t^{\tau'} - r_t P(t, \tau')}{\sigma_t^{\tau'}} = \eta(t, r_t)
  \]

- Then, for all bonds, must have
  \[
  \mu_t^\tau - r_t P(t, \tau) = \eta(t, r_t) \sigma_t^\tau
  \]

- Recall the definition of \( \mu_t^\tau, \sigma_t^\tau \) to derive the pricing PDE on \( P(t, \tau) = f(t, r_t, \tau) \)
  \[
  \frac{\partial f}{\partial t} + \frac{\partial f}{\partial r} \alpha(r) + \frac{1}{2} \frac{\partial^2 f}{\partial r^2} \beta^2(r) - rf = \eta(t, r) \beta(r) \frac{\partial f}{\partial r}, \quad f(\tau, r, \tau) = 1
  \]

- The solution, indeed, has the form \( f(t, r, \tau) \).
What if

\[ \frac{\mu_t^\tau - r_tP(t, \tau)}{\sigma_t^\tau} \neq \eta(t, r_t) \]

for any function \( \eta \), i.e., the LHS depends on something other than \( r_t \) and \( t \)? Then the term structure will not have a single-factor form.

We have seen that, for each choice of \( \eta(t, r_t) \), absence of arbitrage implies that bond prices must satisfy the pricing PDE.

The reverse is true: if bond prices satisfy the pricing PDE (with well-behaved \( \eta(t, r_t) \)), there is no arbitrage (show later, using risk-neutral pricing).

The choice of \( \eta(t, r_t) \) determines the joint arbitrage-free dynamics of bond prices (yields).

\( \eta(t, r_t) \) is the price of interest rate risk.
Multiple Brownian motions

- Consider two independent Brownian motions $Z_t^1$ and $Z_t^2$. Construct a third process
  \[ X_t = \rho Z_t^1 + \sqrt{1 - \rho^2} Z_t^2 \]
- $X_t$ is also a Brownian motion:
  - $X_t$ has IID normal increments;
  - $X_t$ is continuous.
- $X_t$ and $Z_t^1$ are correlated:
  \[ E_0 [X_t Z_t^1] = \rho t \]
- Correlated Brownian motions can be constructed from uncorrelated ones, just like with normal random variables.
- Cross-variation
  \[ [Z_t^1, Z_t^2]_T = \lim_{\Delta \to 0} \sum_{n=1}^{N-1} (Z_t^1(t_{n+1}) - Z_t^1(t_n)) \times (Z_t^2(t_{n+1}) - Z_t^2(t_n)) = 0 \]
- Short-hand rule
  \[ dZ_t^1 \ dZ_t^2 = 0 \implies dZ_t^1 \ dX_t = \rho \ dt \]
Multivariate Itô Processes

- A multivariate Itô process is a vector process with each coordinate driven by an Itô process.
- Consider a pair of processes

\[
\begin{align*}
\mathrm{d}X_t &= \mu^X_t \, \mathrm{d}t + \sigma^X_t \, \mathrm{d}Z^X_t, \\
\mathrm{d}Y_t &= \mu^Y_t \, \mathrm{d}t + \sigma^Y_t \, \mathrm{d}Z^Y_t, \\
\mathrm{d}Z^X_t \, \mathrm{d}Z^Y_t &= \rho_t \, \mathrm{d}t
\end{align*}
\]

- Itô’s formula can be extended to multiple process as follows:

\[
\begin{align*}
\mathrm{d}f(t, X_t, Y_t) &= \frac{\partial f}{\partial t} \, \mathrm{d}t + \frac{\partial f}{\partial X_t} \, \mathrm{d}X_t + \frac{\partial f}{\partial Y_t} \, \mathrm{d}Y_t + \\
&\quad \frac{1}{2} \frac{\partial^2 f}{\partial X_t^2} (\mathrm{d}X_t)^2 + \frac{1}{2} \frac{\partial^2 f}{\partial Y_t^2} (\mathrm{d}Y_t)^2 + \frac{\partial^2 f}{\partial X_t \partial Y_t} \, \mathrm{d}X_t \, \mathrm{d}Y_t
\end{align*}
\]
Example of Itô’s formula

Consider two asset price processes, $X_t$ and $Y_t$, both given by Itô processes

$$dX_t = \mu_t^X \, dt + \sigma_t^X \, dZ_t^X$$
$$dY_t = \mu_t^Y \, dt + \sigma_t^Y \, dZ_t^Y$$
$$dZ_t^X \, dZ_t^Y = 0$$

Using Itô’s formula, we can derive the process for the ratio $f_t = X_t / Y_t$ (use $f(X, Y) = X / Y)$:

$$\frac{df_t}{f_t} = \frac{dX_t}{X_t} - \frac{dY_t}{Y_t} - \frac{dX_t}{X_t} \frac{dY_t}{Y_t} + \left( \frac{dY_t}{Y_t} \right)^2$$

$$= \left( \frac{\mu_t^X}{X_t} - \frac{\mu_t^Y}{Y_t} + \frac{(\sigma_t^Y)^2}{Y_t^2} \right) \, dt + \frac{\sigma_t^X}{X_t} \, dZ_t^X - \frac{\sigma_t^Y}{Y_t} \, dZ_t^Y$$
Example of Itô’s formula

- We find that the expected growth rate of the ratio $X_t/Y_t$ is
  
  $$
  \left( \frac{\mu_t^X}{X_t} - \frac{\mu_t^Y}{Y_t} + \frac{(\sigma_t^Y)^2}{Y_t^2} \right) 
  $$

- Assume that $\mu_t^X = \mu_t^Y$. Then,

  $$
  E_t \left( \frac{df_t}{f_t} \right) = \frac{(\sigma_t^Y)^2}{Y_t^2} \, dt 
  $$

- Repeating the same calculation for the inverse ratio, $h_t = Y_t/X_t$, we find

  $$
  E_t \left( \frac{dh_t}{h_t} \right) = \frac{(\sigma_t^X)^2}{X_t^2} \, dt 
  $$

- It is possible for both the ratio $X_t/Y_t$ and its inverse $Y_t/X_t$ to be expected to grow at the same time. Application to FX.
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Stochastic Differential Equations

Example: Heston’s stochastic volatility model

\[
\frac{dS_t}{S_t} = \mu \, dt + \sqrt{v_t} \, dZ^S_t \\
v_t = -\kappa(v_t - \bar{v}) \, dt + \gamma \sqrt{v_t} \, dZ^v_t \\
dZ^S_t \, dZ^v_t = \rho \, dt
\]

Conditional variance \( v_t \) is described by a Stochastic Differential Equation.

Definition (SDE)

The Itô process \( X_t \) satisfies a stochastic differential equation

\[
dX_t = \mu(t, X_t) \, dt + \sigma(t, X_t) \, dZ_t
\]

with an initial condition \( X_0 \) if it satisfies

\[
X_t = X_0 + \int_0^t \mu(s, X_s) \, ds + \sigma(s, X_s) \, dZ_s.
\]
Existence of Solutions of SDEs

Assume that for some $C, D > 0$

$$|\mu(t, X)| + |\sigma(t, X)| \leq C(1 + |X|)$$

and

$$|\mu(t, X) - \mu(t, Y)| + |\sigma(t, X) - \sigma(t, Y)| \leq D|X - Y|$$

for any $X$ and $Y$ (Lipschitz property).

Then, the SDE

$$dX_t = \mu(t, X_t) \, dt + \sigma(t, X_t) \, dZ_t, \quad X_0 = x,$$

has a unique continuous solution $X_t$. 
Common SDEs

Arithmetic Brownian Motion

The solution of the SDE

\[ dX_t = \mu \, dt + \sigma \, dZ_t \]

is given by

\[ X_t = X_0 + \mu t + \sigma Z_t. \]

The process \( X_t \) is called an arithmetic Brownian motion, or Brownian motion with a drift.

- Guess and verify.
- We typically reduce an SDE to a few common cases with explicit solutions.
Common SDEs

Geometric Brownian Motion

- Consider the SDE
  \[ dX_t = \mu X_t \, dt + \sigma X_t \, dZ_t \]

- Define the process
  \[ Y_t = \ln(X_t) \]

By Itô's Lemma,

\[ dY_t = \frac{1}{X_t} \mu X_t \, dt + \frac{1}{X_t} \sigma X_t \, dZ_t + \frac{1}{2} \left( -\frac{1}{X_t^2} \right) \sigma^2 X_t^2 \, dt = \left( \mu - \frac{\sigma^2}{2} \right) dt + \sigma \, dZ_t \]

- \( Y_t \) is an arithmetic Brownian motion, given in the previous example, and
  \[ Y_t = Y_0 + \left( \mu - \frac{\sigma^2}{2} \right) t + \sigma Z_t \]

- Then
  \[ X_t = e^{Y_t} = X_0 \exp \left( \left( \mu - \frac{\sigma^2}{2} \right) t + \sigma Z_t \right) \]
Common SDEs

Ornstein-Uhlenbeck process

- The mean-reverting Ornstein-Uhlenbeck process is the solution \( X_t \) to the stochastic differential equation

\[
dX_t = (\overline{X} - X_t) \, dt + \sigma \, dZ_t
\]

- We solve this equation using \( e^t \) as an integrating factor.
- Setting \( Y_t = e^t \) and using Itô's lemma for the function \( f(X, Y) = XY \), we find

\[
d(e^t X_t) = e^t(\overline{X} \, dt + \sigma \, dZ_t).
\]

Integrating this between 0 and \( t \), we find

\[
e^t X_t - X_0 = \int_0^t e^s \overline{X} \, ds + \int_0^t e^s \sigma \, dZ_s,
\]

i.e.,

\[
X_t = e^{-t} X_0 + (1 - e^{-t}) \overline{X} + \sigma \int_0^t e^{s-t} \, dZ_s.
\]
Option Replication in the Heston Model

- Assume the Heston stochastic-volatility model for the stock.
- Attempt to replicate the option payoff with the stock and the risk-free bond.
- Can we find a trading strategy that would guarantee perfect replication?
- It is possible to replicate an option using a bond, a stock, and another option.
Recap of APT

Recall the logic of APT.

Suppose we have $N$ assets with two-factor structure in their returns:

$$R_t^i = a_i + b_i^1 F_t^1 + b_i^2 F_t^2$$

Interest rate is $r$.

While not stated explicitly, all factor loadings may be stochastic.

Unlike the general version of the APT, we assume that returns have no idiosyncratic component.

At time $t$, consider any portfolio with fraction $\theta_i$ in each asset $i$ that has zero exposure to both factors:

$$b_i^1 \theta_1 + b_i^2 \theta_2 + \ldots + b_i^N \theta_N = 0$$
$$b_i^1 \theta_1 + b_i^2 \theta_2 + \ldots + b_i^N \theta_N = 0$$

This portfolio must have zero expected excess return to avoid arbitrage

$$\theta_1 \left( E_t[R_t^1] - r \right) + \theta_2 \left( E_t[R_t^2] - r \right) + \ldots + \theta_N \left( E_t[R_t^N] - r \right) = 0$$
Recap of APT

To avoid arbitrage, any portfolio satisfying

\[ b_1^1 \theta_1 + b_2^2 \theta_2 + \ldots + b_1^N \theta_N = 0 \]
\[ b_2^1 \theta_1 + b_2^2 \theta_2 + \ldots + b_2^N \theta_N = 0 \]

must satisfy

\[ \theta_1 \left( E_t[R_t^1] - r \right) + \theta_2 \left( E_t[R_t^2] - r \right) + \ldots + \theta_N \left( E_t[R_t^N] - r \right) = 0 \]

Restating this in vector form, any vector orthogonal to

\((b_1^1, b_1^2, \ldots, b_1^N)\) and \((b_2^1, b_2^2, \ldots, b_2^N)\)

must be orthogonal to \((E_t[R_t^1] - r, E_t[R_t^2] - r, \ldots, E_t[R_t^N] - r)\).

Conclude that the third vector is spanned by the first two: there exist constants (prices of risk) \((\lambda_t^1, \lambda_t^2)\) such that

\[ E_t[R_t^i] - r = \lambda_t^1 b^i_1 + \lambda_t^2 b^i_2, \quad i = 1, \ldots, N \]
Option Pricing in the Heston Model

- Suppose there are \( N \) derivatives with prices given by
  
  \[ f^i(t, S_t, \nu_t) \]

  where the first option is the stock itself: \( f^1(t, S_t, \nu_t) = S_t \).

- Using Ito’s lemma, their prices satisfy

  \[
  df^i(t, S_t, \nu_t) = a^i_t dt + \frac{\partial f^i(t, S_t, \nu_t)}{\partial S_t} dS_t + \frac{\partial f^i(t, S_t, \nu_t)}{\partial \nu_t} d\nu_t
  \]

- Compare the above to our APT argument

- Conclude that there exist \( \lambda^S_t \) and \( \lambda^\nu_t \) such that

  \[
  E \left[ df^i(t, S_t, \nu_t) - rf^i(t, S_t, \nu_t) \right] dt = \frac{\partial f^i(t, S_t, \nu_t)}{\partial S_t} \lambda^S_t dt + \frac{\partial f^i(t, S_t, \nu_t)}{\partial \nu_t} \lambda^\nu_t dt
  \]

- We work with price changes instead of returns, as we did in the APT, because some of the derivatives may have zero price.
Option Pricing in the Heston Model

- The APT pricing equation, applied to the stock, implies that

$$E [dS_t - rS_t\ dt] = (\mu - r)S_t\ dt = \lambda^S_t\ dt$$

- $\lambda^v_t$ is the price of volatility risk, which determines the risk premium on any investment with exposure to $dv_t$.

- Writing out the pricing equation explicitly, with the Ito’s lemma providing an expression for $E [df^i(t, S_t, v_t)]$,

$$\frac{\partial f^i}{\partial t} + \frac{\partial f^i}{\partial S} \mu S + \frac{\partial f^i}{\partial v} (-\kappa)(v - \bar{v}) + \frac{1}{2} \frac{\partial^2 f^i}{\partial S^2} v S^2 + \frac{1}{2} \frac{\partial^2 f^i}{\partial v^2} \gamma^2 v + \frac{\partial^2 f^i}{\partial S \partial v} \rho \gamma S v - rf^i = \frac{\partial f^i}{\partial S} (\mu - r) S + \frac{\partial f^i}{\partial v} \lambda^v_t$$

- As long as we assume that the price of volatility risk is of the form

$$\lambda^v_t = \lambda^v(t, S_t, v_t)$$

the assumed functional form for option prices is justified and we obtain an arbitrage-free option pricing model.
Numerical Solution of SDEs

- Except for a few special cases, SDEs do not have explicit solutions.
- The most basic and common method of approximating solutions of SDEs numerically is using the first-order Euler scheme.
- Use the grid \( t_i = i \Delta \).

\[
\hat{X}_{i+1} = \hat{X}_i + \mu(t_i, \hat{X}_i) \Delta + \sigma(t_i, \hat{X}_i) \sqrt{\Delta} \tilde{\varepsilon}_i,
\]

where \( \tilde{\varepsilon}_i \) are IID \( \mathcal{N}(0, 1) \) random variables.

- Using a binomial distribution for \( \tilde{\varepsilon}_i \), with equal probabilities of \( \pm 1 \), is also a valid procedure for approximating the distribution of \( X_t \).
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Moments of Diffusion Processes

- Often need to compute conditional moments of diffusion processes:
  - Expected returns and variances of returns on financial assets over finite time intervals;
  - Use the method of moments to estimate a diffusion process from discretely sampled data;
  - Compute prices of derivatives.

- One approach is to reduce the problem to a PDE, which can sometimes be solved analytically.
Kolmogorov Backward Equation

- Diffusion process $X_t$ with coefficients $\mu(t, X)$ and $\sigma(t, X)$.
- Objective: compute a conditional expectation

$$f(t, X) = E[g(X_T) | X_t = X]$$

- Suppose $f(t, X)$ is a smooth function of $t$ and $X$. By the law of iterated expectations,

$$f(t, X_t) = E_t[f(t + dt, X_{t+dt})] \Rightarrow E_t[df(t, X_t)] = 0$$

- Using Ito’s Lemma,

$$E_t[df(t, X_t)] = \left( \frac{\partial f(t, X)}{\partial t} + \mu(t, X) \frac{\partial f(t, X)}{\partial X} + \frac{1}{2} \sigma(t, X)^2 \frac{\partial^2 f(t, X)}{\partial X^2} \right) dt = 0$$

- Kolmogorov backward equation

$$\frac{\partial f(t, X)}{\partial t} + \mu(t, X) \frac{\partial f(t, X)}{\partial X} + \frac{1}{2} \sigma(t, X)^2 \frac{\partial^2 f(t, X)}{\partial X^2} = 0,$$

with boundary condition

$$f(T, X) = g(X)$$
Example: Square-Root Diffusion

- Consider a popular diffusion process used to model interest rates and stochastic volatility

\[ dX_t = -\kappa(X_t - \bar{X}) \, dt + \sigma \sqrt{X_t} \, dZ_t \]

- We want to compute the conditional moments of this process, to be used as a part of GMM estimation.

- Compute the second non-central moment

\[ f(t, X) = E(X_t^2 | X_t = X) \]

- Using Kolmogorov backward equation,

\[
\frac{\partial f(t, X)}{\partial t} - \kappa(X - \bar{X}) \frac{\partial f(t, X)}{\partial X} + \frac{1}{2} \sigma^2 X \frac{\partial^2 f(t, X)}{\partial X^2} = 0,
\]

with boundary condition

\[ f(T, X) = X^2 \]
Example: Square-Root Diffusion

- Look for the solution in the form
  \[ f(t, X) = a_0(t) + a_1(t)X + \frac{a_2(t)}{2}X^2 \]

- Substitute \( f(t, X) \) into the PDE
  \[
  a'_0(t) + a'_1(t)X + \frac{a'_2(t)}{2}X^2 - \kappa(X - \overline{X})(a_1(t) + a_2(t)X) + \frac{\sigma^2}{2}Xa_2(t) = 0
  \]

- Collect terms with different powers of \( X \), zero, one and two, to get
  \[
  a'_0(t) + \kappa \overline{X}a_1(t) = 0
  \]
  \[
  a'_1(t) - \kappa a_1(t) + \left( \frac{\sigma^2}{2} + \kappa \overline{X} \right) a_2(t) = 0
  \]
  \[
  a'_2(t) - 2\kappa a_2(t) = 0
  \]

with initial conditions
\[
 a_0(T) = a_1(T) = 0, \quad a_2(T) = 2
\]
Example: Square-Root Diffusion

We solve the system of equations starting from the third one and working up to the first:

\[
\begin{align*}
a_0(t) &= \overline{X} A \left( \frac{1}{2} e^{2\kappa(t-T)} - e^{\kappa(t-T)} + \frac{1}{2} \right) \\
a_1(t) &= A \left( e^{\kappa(t-T)} - e^{2\kappa(t-T)} \right) \\
a_2(t) &= 2e^{2\kappa(t-T)} \\
A &= \frac{\sigma^2 + 2\kappa \overline{X}}{\kappa}
\end{align*}
\]

Compare the exact expression above to an approximate expression, obtained by assuming that \(T - t = \Delta t\) is small.
Example: Square-Root Diffusion

- Assume $T - t = \Delta t$ is small. Using Taylor expansion,

$$a_0(t) = o(\Delta t), \quad a_1(t) = A\kappa \Delta t + o(\Delta t)$$
$$a_2(t) = 2(1 - 2\kappa \Delta t) + o(\Delta t)$$

Then

$$E_t(X_{t+\Delta t}^2 | X_t = X) \approx X^2 + \Delta t \left( (\sigma^2 + 2\kappa \bar{X}) X - 2\kappa X^2 \right)$$

- Alternatively,

$$X_{t+\Delta t} \approx X_t - \kappa (X_t - \bar{X}) \Delta t + \sigma \sqrt{X_t} \sqrt{\Delta t} \varepsilon_t, \quad \varepsilon_t \sim \mathcal{N}(0, 1)$$

Therefore

$$E_t(X_{t+\Delta t}^2 | X_t = X) \approx X^2 + \Delta t \left( \sigma^2 X - 2\kappa X(X - \bar{X}) \right)$$
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Risk-Neutral Probability Measure

- Under the risk-neutral probability measure $Q$, expected conditional asset returns must equal the risk-free rate.

- Alternatively, using the discounted cash flow formula, the price $P_t$ of an asset with payoff $H_T$ at time $T$ is given by

$$P_t = E_t^Q \left[ \exp \left( - \int_t^T r_s \, ds \right) H_T \right]$$

where $r_s$ is the instantaneous risk-free interest rate at time $s$.

- The DCF formula under the risk-neutral probability can be used to compute asset prices by Monte Carlo simulation (e.g., using an Euler scheme to approximate the solutions of SDEs).

- Alternatively, one can derive a PDE characterizing asset prices using the connection between PDEs and SDEs.
Change of Measure

- We often want to consider a probability measure $Q$ different from $P$, but the one that agrees with $P$ on which events have zero probability (equivalent to $P$) (e.g., $Q$ could be a risk-neutral measure).
- Different probability measures assign different relative likelihoods to the trajectories of the Brownian motion.
- It is easy to express a new probability measure $Q$ using its density

$$\xi_T = \left( \frac{dQ}{dP} \right)_T$$

- For any random variable $X_T$,

$$E_0^Q[X_T] = E_0^P[\xi_T X_T], \quad E_t^Q[X_T] = E_t^P \left[ \frac{\xi_T}{\xi_t} X_T \right], \quad \xi_t \equiv E_t^P[\xi_T]$$

- $Q$ is equivalent to $P$ if $\xi_T$ is positive (with probability one).
Change of Measure

- If we consider a probability measure $Q$ different from $P$, but the one that agrees with $P$ on which events have zero probability, then the $P$-Brownian motion $Z_t^P$ becomes an Ito process under $Q$:

$$dZ_t^P = dZ_t^Q - \eta_t \, dt$$

for some $\eta_t$. $Z_t^Q$ is a Brownian motion under $Q$.

- When we change probability measures this way, only the drift of the Brownian motion changes, not the variance.

- Intuition: a probability measure assigns relative likelihood to different trajectories of the Brownian motion. Variance of the Ito process can be recovered from the shape of a single trajectory (quadratic variation), so it does not depend on the relative likelihood of the trajectories, hence, does not depend on the choice of the probability measure.
Risk-Neutral Probability Measure

- Under the risk-neutral probability measure, expected conditional asset returns must equal the risk-free rate.
- Start with the stock price process under \( P \):
  \[
  dS_t = \mu_t S_t \, dt + \sigma_t S_t \, dZ_t^P
  \]

- Under the risk-neutral measure \( Q \),
  \[
  dS_t = r_t S_t \, dt + \sigma_t S_t \, dZ_t^Q
  \]

Thus, if \( dZ_t^P = -\eta_t \, dt + dZ_t^Q \),

\[
\mu_t - \sigma_t \eta_t = r_t
\]

\( \eta_t \) is the price of risk.

- The risk-neutral measure \( Q \) is such that the process \( Z_t^Q \) defined by
  \[
  dZ_t^Q = \frac{\mu_t - r_t}{\sigma_t} \, dt + dZ_t^P
  \]

is a Brownian motion under \( Q \).
Under the risk-neutral probability measure, expected conditional asset returns must equal the risk-free rate.

We conclude that for any asset (paying no dividends), the conditional risk premium is given by

$$E^P_t \left[ \frac{dS_t}{S_t} \right] - r_t \, dt = E^P_t \left[ \frac{dS_t}{S_t} \right] - E^Q_t \left[ \frac{dS_t}{S_t} \right]$$

Thus, mathematically, the risk premium is the difference between expected returns under the $P$ and $Q$ probabilities.
Risk-Neutral Probability Measure

- If we want to connect \( Q \) to \( P \) explicitly, how can we compute the density, \( dQ/dP \)?
- In discrete time, the density was conditionally lognormal.
- The density \( \xi_T = (dQ/dP)_T \) is given by

  \[
  \xi_t = \exp \left( - \int_0^t \eta_u \, dZ_u^P - \frac{1}{2} \int_0^t \eta_u^2 \, du \right), \quad 0 \leq t \leq T
  \]

- The state-price density is given by

  \[
  \pi_t = \exp \left( \int_0^t -r_u \, du \right) \xi_t
  \]

- The reverse is true: if we define \( \xi_T \) as above, for any process \( \eta_t \) satisfying certain regularity conditions (e.g., \( \eta_t \) is bounded, or satisfies the Novikov’s condition as in Back 2005, Appendix B.1), then measure \( Q \) is equivalent to \( P \) and

  \[
  Z_t^Q = Z_t^P + \int_0^t \eta_u \, du
  \]

  is a Brownian motion under \( Q \).
Risk-Neutral Probability and Arbitrage

- If there exists a risk-neutral probability measure, then the model is arbitrage-free.
- If there exists a **unique** risk-neutral probability measure in a model, then all options are redundant and can be replicated by trading in the underlying assets and the risk-free bond.
- A convenient way to build arbitrage-free models is to describe them directly under the risk-neutral probability.
- One does not need to describe the $P$ measure explicitly to specify an arbitrage-free model.
- However, to estimate models using historical data, particularly, to estimate risk premia, one must specify the price of risk, i.e., the link between $Q$ and $P$. 
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Black-Scholes Model

- Assume that the stock pays no dividends and the stock price follows

\[
\frac{dS_t}{S_t} = \mu \, dt + \sigma \, dZ^P_t
\]

- Assume that the interest rate is constant, \( r \).

- Under the risk-neutral probability \( Q \), the stock price process is

\[
\frac{dS_t}{S_t} = r \, dt + \sigma \, dZ^Q_t
\]

Terminal stock price \( S_T \) is lognormally distributed:

\[
\ln S_T = \ln S_0 + \left( r - \frac{\sigma^2}{2} \right) T + \sigma \sqrt{T} \varepsilon^Q, \quad \varepsilon^Q \sim \mathcal{N}(0, 1)
\]

- Price of any European option with payoff \( H(S_T) \) can be computed as

\[
P_t = \mathbb{E}_t^Q \left[ e^{-r(T-t)} H(S_T) \right]
\]
Consider the Vasicek model of bond prices.

A single-factor arbitrage-free model.

To guarantee that the model is arbitrage-free, build it under the risk-neutral probability measure.

Assume the short-term risk-free rate process under $\mathcal{Q}$

$$dr_t = -\kappa (r_t - \bar{r}) \, dt + \sigma \, dZ_t^\mathcal{Q}$$

Price of a pure discount bond maturing at $T$ is given by

$$P(t, T) = E_t^\mathcal{Q} \left[ \exp \left( - \int_t^T r_s \, ds \right) \right]$$

Characterize $P(t, T)$ as a solution of a PDE.
Term Structure of Interest Rates

- Look for $P(t, T) = f(t, r_t)$.
- Using Ito’s lemma,

$$
E_t^Q[df(t, r_t)] = \left( \frac{\partial f(t, r_t)}{\partial t} - \kappa (r_t - \bar{r}) \frac{\partial f(t, r_t)}{\partial r_t} + \frac{1}{2} \sigma^2 \frac{\partial^2 f(t, r_t)}{\partial r_t^2} \right) \, dt
$$

- Risk-neutral pricing requires that

$$
E_t^Q[df(t, r_t)] = r_t f(t, r_t) \, dt
$$

and therefore $f(t, r_t)$ must satisfy the PDE

$$
\frac{\partial f(t, r)}{\partial t} - \kappa (r - \bar{r}) \frac{\partial f(t, r)}{\partial r} + \frac{1}{2} \sigma^2 \frac{\partial^2 f(t, r)}{\partial r^2} = rf(t, r)
$$

with the boundary condition

$$
f(T, r) = 1
$$
Term Structure of Interest Rates

- Look for the solution in the form

\[ f(t, r_t) = \exp(-a(T - t) - b(T - t)r_t) \]

- Derive a system of ODEs on \( a(t) \) and \( b(t) \) to find

\[
\begin{align*}
    a(T - t) &= \bar{r}(T - t) - \frac{\bar{r}}{\kappa} \left(1 - e^{-\kappa(T-t)}\right) - \\
    &\quad \frac{\sigma^2}{4\kappa^3} \left(2\kappa(T - t) - e^{-2\kappa(T-t)} + 4e^{-\kappa(T-t)} - 3\right) \\
    b(T - t) &= \frac{1}{\kappa} \left(1 - e^{-\kappa(T-t)}\right)
\end{align*}
\]
Term Structure of Interest Rates

- Assume a constant price of risk $\eta$. What does this imply for the interest rate process under the physical measure $\mathbb{P}$ and for the bond risk premia?

- Use the relation

$$dZ_t^\mathbb{P} = -\eta \, dt + dZ_t^\mathbb{Q}$$

to derive

$$dr_t = -\kappa (r_t - \bar{r}) \, dt + \sigma \eta \, dt + \sigma dZ_t^\mathbb{P} = -\kappa \left( r_t - \left( \bar{r} + \frac{\sigma \eta}{\kappa} \right) \right) \, dt + \sigma dZ_t^\mathbb{P}$$

- Expected bond returns satisfy

$$\mathbb{E}_t^\mathbb{P} \left( \frac{dP(t, T)}{P(t, T)} \right) = (r_t + \sigma_t^\mathbb{P} \eta) \, dt$$

where

$$\sigma_t^\mathbb{P} = \frac{1}{f(t, r_t)} \frac{\partial f(t, r_t)}{\partial r_t} \sigma = -b(T - t) \sigma$$

- $|b(T - t) \sigma|$ is the volatility of bond returns.
Consider again the Heston’s model. Assume that under the risk-neutral probability $Q$, stock price is given by

$$d \ln S_t = (r - \frac{1}{2} \nu_t) \, dt + \sqrt{\nu_t} dZ^{Q,S}_t$$

$$d\nu_t = -\kappa (\nu_t - \bar{\nu}) \, dt + \gamma \rho \sqrt{\nu_t} dZ^{Q,S}_t + \gamma \sqrt{1 - \rho^2} \sqrt{\nu_t} dZ^{Q,v}_t$$

$$dZ^{Q,S}_t \, dZ^{Q,v}_t = 0$$

$Z^{Q,v}_t$ models volatility shocks uncorrelated with stock returns.

Constant interest rate $r$.

The price of a European option with a payoff $H(S_T)$ can be computed as

$$P_t = E^Q_t \left[ \exp \left( -r (T - t) \right) H(S_T) \right]$$

We haven’t said anything about the physical process for stochastic volatility. In particular, how is volatility risk priced?
Equity Options with Stochastic Volatility

In the Heston's model, assume that the price of volatility risk is constant, $\eta^v$, and the price of stock price risk is constant, $\eta^S$.

Then, under $\mathbb{P}$, stock returns follow

$$d \ln S_t = \left( r + \eta^S \sqrt{v_t} - \frac{1}{2} v_t \right) dt + \sqrt{v_t} dZ_t^{P,S}$$

$$dv_t = \left( -\kappa (v_t - \bar{v}) + \gamma \sqrt{v_t} \left( \rho \eta^S + \sqrt{1 - \rho^2} \eta^v \right) \right) dt + \gamma \rho \sqrt{v_t} dZ_t^{P,S} + \gamma \sqrt{1 - \rho^2} \sqrt{v_t} dZ_t^{P,v}$$

$$dZ_t^{P,s} dZ_t^{P,v} = 0$$

We have used

$$dZ_t^{P,S} = -\eta^S dt + dZ_t^{Q,S}$$

$$dZ_t^{P,v} = -\eta^v dt + dZ_t^{Q,v}$$

Conditional expected excess stock return is

$$E_t^P \left[ \frac{dS_t}{S_t} - r dt \right] = E_t^P \left[ d \ln S_t + \frac{v_t}{2} dt - r dt \right] = \left( \eta^S \sqrt{v_t} \right) dt$$

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Our assumptions regarding the market prices of risk translate directly into implications for return predictability.

For stock returns, our assumption of constant price of risk predicts a nonlinear pattern in excess returns: expected excess stock returns proportional to conditional volatility.

Suppose we construct a position in options with the exposure $\lambda_t$ to stochastic volatility shocks and no exposure to the stock price:

$$dW_t = [...] \, dt + \lambda_t dZ_t^{P,v}$$

Then the conditional expected gain on such a position is

$$(W_t r + \lambda_t \eta^v) \, dt$$
Heston’s Model of Stochastic Volatility

- Assume that under the risk-neutral probability $\mathbb{Q}$, stock price is given by

$$d \ln S_t = \left( r - \frac{1}{2} v_t \right) dt + \sqrt{v_t} dZ^Q_t, S$$

$$d v_t = -\kappa (v_t - \overline{v}) dt + \gamma \rho \sqrt{v_t} dZ^Q_t, S + \gamma \sqrt{1 - \rho^2} \sqrt{v_t} dZ^Q_t, V$$

$$dZ^Q_t, S \ dZ^Q_t, V = 0$$

- $Z^Q_t, V$ models volatility shocks uncorrelated with stock returns.
- Constant interest rate $r$.
- The price of a European option with a payoff $H(S_T)$ can be computed as

$$P_t = E_t^Q [\exp (-r (T - t)) H(S_T)]$$

- Assume that the price of volatility risk is constant, $\eta^V$, and the price of stock price risk is constant, $\eta^S$. 
Variance Swap in Heston’s Model

- Consider a variance swap, paying

\[
\int_t^T (d \ln S_u)^2 - K_t^2
\]

at time \(T\). What should be the strike price of the swap, \(K_t\), to make sure that the market value of the swap at time \(t\) is zero?

- Using the result on quadratic variation,

\[
(d \ln S_t)^2 = v_t \, dt
\]

the strike price must be such that

\[
e^{-r(T-t)} E_t^Q \left[ \left( \int_t^T v_u \, du - K_t^2 \right) \right] = 0
\]

- Need to compute

\[
K_t^2 = E_t^Q \left[ \int_t^T v_u \, du \right]
\]
Variance Swap in Heston’s Model

Since

\[ v_u = v_t - \int_t^u \kappa (v_s - \bar{v}) \, ds + \gamma \rho \int_t^u \sqrt{v_s} \, dZ^{Q,S}_s + \gamma \sqrt{1 - \rho^2} \int_t^u \sqrt{v_s} \, dZ^{Q,V}_s \]

we find that

\[ E^Q_t[v_u] = v_t - E^Q_t \left[ \int_t^u \kappa (v_s - \bar{v}) \, ds \right] = v_t - \int_t^u \kappa (E^Q_t[v_s] - \bar{v}) \, ds \]

Solving the above equation for \( E^Q_t[v_u] \), we find

\[ E^Q_t[v_u] = \bar{v} + e^{-\kappa (u-t)} (v_t - \bar{v}) \]

We obtain the strike price

\[ K^2_t = E^Q_t \left[ \int_t^T v_u \, du \right] = \bar{v} (T - t) + (v_t - \bar{v}) \frac{1}{\kappa} \left( 1 - e^{-\kappa (T-t)} \right) \]
Expected Profit/Loss on a Variance Swap

To compute expected profit/loss on a variance swap, we need to evaluate

$$E_t^P \left[ \int_t^T (d \ln S_u)^2 - K_t^2 \right]$$

Instantaneous expected gain on a long position in the swap is easy to compute in closed form.

The market value of a swap starts at 0 at time $t$, and at $s > t$ becomes

$$P_s \equiv E_s^Q \left[ e^{-r(T-s)} \left( \int_t^S v_u \, du + \int_T^S v_u \, du - K_t^2 \right) \right]$$

$$= e^{-r(T-s)} \left( \int_t^S v_u \, du + K_s^2 - K_t^2 \right)$$

We conclude that the instantaneous gain on the long swap position at time $s$ equals

$$[...] \, ds + \frac{1}{\kappa} e^{-r(T-s)} \left( 1 - e^{-\kappa(T-s)} \right) \, dv_s$$
Expected Profit/Loss on a Variance Swap

- We conclude that the instantaneous gain on the long swap position at time $s$ equals
  
  $$\left[\ldots\right] ds + \frac{1}{\kappa} e^{-r(T-s)} \left(1 - e^{-\kappa(T-s)}\right) d\nu_s$$

- Given our assumed market prices of risk, $\eta^S$ and $\eta^v$, the time-$s$ expected instantaneous gain on the swap opened at time $t$ is

  $$P_s r \ ds + \gamma \sqrt{\nu_s} \frac{1}{\kappa} e^{-r(T-s)} \left(1 - e^{-\kappa(T-s)}\right) \left(\rho \eta^S + \sqrt{1 - \rho^2 \eta^v}\right) ds$$
Summary

- Risk-neutral pricing is a convenient framework for developing arbitrage-free pricing models.
- Connection to classical results: risk-neutral expectation can be characterized by a PDE.
- Risk premium on an asset is the difference between expected return under $\mathbb{P}$ and under $\mathbb{Q}$ probability measures.
Readings
