Analysis of a Forecasting-Production-Inventory System with Stationary Demand

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Forecasting-Production-Inventory

- Make-to-Stock environment

\[ P_t = \min \{ Q_{t-1}, C_t \} \]
Objective

- Minimize steady-state
  - Inventory holding costs $h$
  - Shortage penalty costs $b$
Recap: MMFE Model

- Rolling horizon $H$
- Forecast $D_{t,t+i}$ $i = 0,\ldots,H$
- Forecast Update

$$
\varepsilon_{t,t+i} = D_{t,t+i} - D_{t-1,t+i}
$$

- Assumptions
  - Stationary Demand with rate $\lambda$
  - Unbiased Forecasts
  - Uncorrelated Forecast Updates
Production-Inventory Model

- MRP-Type Release Policy:
  \[ R_t = \sum_{i=0}^{H-1} \varepsilon_{t,t+i} + D_{t,t+H} = e^T \varepsilon_t + \lambda \]

- Inventory Policy
  \[ Q_t + I_t - \sum_{i=1}^{H} D_{t,t+i} = s_H \bigg\{ \tilde{I}_t \bigg\} \]
Production Policy

- Forecast-corrected base-stock policy

\[
P^*(\tilde{I}_{t-1}) = \begin{cases} 
C_t & \text{if } s_H > \tilde{I}_{t-1} + C_t \\
 s_H - \tilde{I}_{t-1} & \text{if } s_H \leq \tilde{I}_{t-1} + C_t
\end{cases}
\]

- State-dependent Optimal Policy

\[(D_{t-1,t}, D_{t-1,t+1}, \ldots, D_{t-1,t+H-1}, \lambda)\]
Benchmark: Myopic Policy

- Do not use available forecast information

\[ R_t = D_t \]

- Constant Inventory

\[ Q_t + I_t = s_m \]
Outline

- Model
- Steady-State Distribution of WIP
- Base-Stock Levels
- Discussion
- Conclusion
In Heavy Traffic, the WIP has an exponential distribution.

Average Excess Capacity

\[ v = \frac{2(\mu - \lambda)}{e^T \Sigma e + \sigma_C^2} \]

Variance of the forecasts

Variance of the production
WIP at time $n$

$$X_n = \sum_{t=1}^{n} (R_t - C_t)$$

$$Q_n = X_n - \inf_{1 \leq t \leq n} X_t$$
Heavy traffic analysis 1

Consider a sequence of systems \( k \) s.t.:

\[
\lambda^{(k)} \rightarrow \lambda \\
\mu^{(k)} \rightarrow \mu \\
\sqrt{k} (\lambda^{(k)} - \mu^{(k)}) \rightarrow c < 0
\]
Heavy traffic analysis 2

\[ \frac{Q_n^{(k)}}{\sqrt{k}} = \frac{X_n^{(k)}}{\sqrt{k}} + \frac{- \inf_{1 \leq t \leq n} X_t^{(k)}}{\sqrt{k}} \]

\[ \frac{X_{[kt]}^{(k)}}{\sqrt{k}} = \frac{X_{[kt]}^{(k)} - m^{(k)} \lfloor kt \rfloor}{\sqrt{k}} + \frac{m^{(k)} \lfloor kt \rfloor}{\sqrt{k}} \]

where \( m^{(k)} = \lambda^{(k)} - \mu^{(k)} \)
Heavy traffic analysis 2

\[
\frac{X^{(k)}_{[kt]}}{\sqrt{k}} = \frac{X^{(k)}_{[kt]} - m^{(k)}_{[kt]}}{\sqrt{k}} + \frac{m^{(k)}_{[kt]}}{\sqrt{k}} + \sigma B(t) + ct
\]

\[BM(c, \sigma^2)\]
Reflected Brownian Motion on the nonnegative halfline

Estimated by an exponential random variable
Steady-state WIP distribution

\[
P(Q_\infty = 0) = 1 - e^{-v\beta}
\]

\[
P(Q_\infty > x) = e^{-vx}(x + \beta)
\]

- \(\beta\) is a correction term, coming from Random Walks
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Determination of the Base-Stock levels 1

- Myopic Policy
  
  \[ s_m = F_{Q_\infty}^{-1}\left(\frac{b}{b + h}\right) \]

- Newsboy quantity…

- … considering the distr. of the WIP!
  
  \[ s_m = \frac{1}{v} \ln\left(1 + \frac{b}{h}\right) - \beta \]
Determination of the Base-Stock levels 2

- MRP-type Policy

\[ s_H = F_W^{-1} \left( \frac{b}{b + h} \right) \]

\[ W = \max \{ Q_\infty + Y_0, \max_{1 \leq k \leq H} Y_k \} \]

- \( Y_0 \) is the difference between:
  - Total Forecast Error over the horizon \( H \) and
  - Total Capacity
Determination of the Base-Stock levels 3

- MRP-type policy: asymptotic
  \[ b \gg h \]
  \[ s^a_H = s^*_m + \mu_{Y_0} + \left( \frac{1}{2} \right) \sigma^2_{Y_0} \nu \]
  Proportional to the variance!

- Good approximation when
  - \( b / h \) large
  - High utilization rate
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Capacity – Stock Trade-Off

- No advance information
  \[ s_h^a = s_m^* - H \lambda \]

- Full advance information
  \[ s_h^a = s_m^* - H \lambda - H (\mu - \lambda) \left( \frac{\sigma_D^2}{\sigma_D^2 + \sigma_C^2} \right) \]

- Interchangeability Capacity/Safety Stock
- Demand variability → Capacity
Discussion

- Correlation $\uparrow \rightarrow \nu \downarrow \rightarrow S_m^* \uparrow$
- $\sigma_{Y_0}^2$ is the system variability over $H$ not resolved at the beginning of the horizon
  - Preference for accurate early forecasts
- Optimize over all planning horizons
  $$R_t = \sum_{i=0}^{H-1} \varepsilon_{t,t+1} + D_{t,t+H}$$
- Greater costs due to
  - misspecification of the forecast model than
  - misuse of the information in production
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Conclusion

- Integrated view
  - Forecast
  - Production
  - Inventory
- Lots of improvement for current MRP systems
- Is heavy traffic of practical value?
Thank you

Questions?