Order Full Rate, Leadtime Variability, and Advance Demand Information in an Assemble-To-Order System

by Lu, Song, and Yao (2002)

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Assembly-to-order system
- Each product is assembled from a set of components,
- Demand for products following batch Poisson processes,
- Inventory of each component follows a base-stock policy
- Replenishment leadtime i.i.d. random variables for each component.

Model as a $M^X / G / \infty$ queue, driven by a common multiclass batch Poisson input stream
- Derive the joint queue-length distribution,
- Order fulfillment performance measure.
Model

- $M$ different components, and $F = \{1, 2, \ldots, m\}$ are the component indices.
- Customer orders arrive as a stationary Poisson process, \(\{A(t), \ t\geq0\}\), with rate $\lambda$.
- Order type $K$: it contains positive units of component in $K$ and 0 units in $F \setminus K$.
  - An order is of type $K$ with probability $q^K$, \(\sum_K q^K = 1\)
  - Type K order stream forms a compound Poisson process with rate $\lambda^K = q^K \lambda$
  - A type-K order has $Q^K_j$ units for each component $j$, $Q_K = (Q^K_j, j \in K)$ has a known discrete distribution.
- For each component $i$, the demand process forms a compound Poisson process.
Model

- Demand are filled on a FCFS basis.
- Demand are backlogged (if one or more components are missing), and are filled on a FCFS basis.
- Inventory of each component is controlled by an independent base-stock policy,
  - Where $s_i$ is the base-stock level for component $i$
  - For each component $i$, replenishment leadtimes, $L_i$, are i.i.d., with a cdf of $G_i$
  - Net inventory at time $t$, $I_i(t) = s_i - X_i(t)$, $i = 1, \ldots, m$, where $X_i(t)$ is the number of outstanding orders of component $i$ at time $t$.
- Immediate availability of all components needed for an arriving demand as the “off-the-shelf” fill rate.
  - Off-the-shelf fill rate of component $i$, $f_i = P[X_i + Q_i \leq s_i]$
  - Off-the-shelf fill rate of demand type $K$, $f^K = P[X_i + Q^K_i \leq s_i, \forall i \in K]$
  - Average (over all demand types) off-the-shelf fill rate, $\bar{f} = \sum_K q^K f^K$
Performance Analysis

• Derive the joint distribution and steady state limit of vector $X(t) = (X_1(t), \ldots, X_m(t))$

(See “Suppliers/Arrivals Replenishment Orders” diagram in Lu, Song, and Yao paper)

– Each component $i$, the number of outstanding orders is exactly the number of jobs in service in an $M^Q_i / G_i / \infty$ queue with Poisson arrival $\lambda_i$ and batch size $Q_i$
– The $m$ queues are not independent.
– Given the number of demand arrivals up to $t$, the $X_i(t)$s are independent of one another.
Performance Analysis

• Proposition 1: $X(t) = (X_1(t), \ldots, X_m(t))$ has a limiting distribution. Derive the generating function of $X$.

• In the special case of unit arrival, $Q_i \equiv 1$, the generating function of $X$ corresponds to a multivariate Poisson distribution. For each $i$, $X_i$ is a Poisson variable with parameter $\lambda_i \ell_i = (\Sigma_{k \epsilon \mathbb{N}_i} \lambda^K) \ell_i$

• The correlation of the queue is solely induced by the common arrivals. If the proportion of the demand types that require both $i$ and $j$ are very small, the correlation between $X_i$ and $X_j$ is negligible.

• Level of correlation is independent of the demand rate.

• Reducing the variability of leadtime or batch sizes will result in a higher correlation among the queue lengths of outstanding jobs.
Response-time-based order fill rate

1) $f^K(w)$ is the probability of having all the components ready within $w$ units of time.
2) $D_i(t, t+u) := D_i(t+u) - D_i(t)$
3) Total number of departures from queue $i$ in $(\tau, \tau+w) = X_i(\tau) + D_i(\tau, \tau+w) - X_i(\tau+w)$
4) $I_i(\tau) + \{X_i(\tau) + D_i(\tau, \tau+w) - X_i(\tau+w)\} \geq 0$
5) $X_i(\tau+w) - D_i(\tau, \tau+w) \leq s_i, \ i \in K$

$$X_i(\tau+w) = X_i^w(\tau) + \sum_{n=1}^{Q^K} 1\{L_i^n > w\} + X_i(\tau+w)$$

6) Demand at $\tau$ can be supplied by $\tau+w$ iff

$$X_i^w(\tau) + X_i(\tau, \tau+w) - D_i(\tau, \tau+w) \leq s_i - \sum_{n=1}^{Q^K} 1\{L_i^n > w\}, \ i \in K$$

$$Y_i := X_i^w - Y_i^w$$

7) Order fill rate of type-K demand within time window $w$, $f^K(w) = P\left[ Y_i + \sum_{n=1}^{Q^K} 1\{L_i^n > w\} \leq s_i, \ \forall i \in K \right]$}

8) Mean:

$$E[Y_i] = \left( \sum_{3 \in \mathcal{R}_i} \lambda^3 E(Q_i^3) \right) (\ell_i - w)$$

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Connection to advance demand information

- Suppose each order arrival epoch is known \( w \) time units in advance, where \( w > 0 \) is a deterministic constant.
- Suppose a type-\( K \) order arrives at \( \tau \), and this information is known at \( \tau + w \), we can fill this order upon its arrival with probability:

\[
 f^K_A(0) = \mathbb{P}\left\{ X_i^w(\tau - w) + X_i(\tau - w, \tau] - D_i(\tau - w, \tau] \leq s_i - \sum_{n=1}^{Q_i^K} \mathbb{1}[L_i^n > w], i \in K \right\} = f^K(w).
\]

- Advance demand information improves the off-the-shelf fill rate:

\[
 f^K(w) \geq f^K(0)
\]

- Compare \( f^K_A(0) \) with that of the modified system, \( \hat{f}^K(0) \), where leadtime is reduced from \( L_i \) to \( \hat{L}_i = [L_i - w]^+ \): \[
 \hat{f}^K(0) = \mathbb{P}\left\{ X_i^w + \sum_{n=0}^{Q_i^K} \mathbb{1}[\hat{L}_i^n > 0] \leq s_i, \forall i \in K \right\} \leq f^K(w)
\]

- Knowing demand in advance (by \( w \) time units) is more effective, in terms of order fill rate, than reducing the supply leadtime of components.