Optimal Control of High-Volume Assemble-to-Order Systems

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Motivation

• Assembly-to-Order
  – hold component inventories
  – rapid assembly of many products
  – Dell - grown by 40% per year in recent years. PC industry - grown by less than 20% per year.

• Challenges of ATO
  – product prices?
  – production capacity for component (supply contract)?
  – dynamically ration scarce components to customer orders?
Overview

• Literature review

• Model formulation
  – Dynamic control problem
  – Static formulation

• Asymptotic analysis

• Delay bound and expediting component option
Literature

• ATO survey by Song and Zipkin (2001)

• not FIFO assembly

• one component and multi-product assembly sequencing — multi-class, single-server queue

• fill rate constraints
Model Formulation

**Sequence of events:**
1. set product prices, component production rates – remain fixed throughout time horizon
2. dynamically sequence assembly of outstanding product orders

**Objective:**
minimize infinite horizon discounted expected profit

**Trade-off:**
inventory vs. customer service (assembly delay, cash flow)

**Operational Assumptions:**
– assembly is instantaneous given necessary components
– customer order for each product are filled FIFO
Model Formulation - notations

\( J \) components
\( K \) finished products
\( a_{kj} \) no. of type \( j \) components needed by product \( k \)
\( p_k \) product price
\( \gamma_j \) component production rate
\( O_k \) product demand arrival renewal process, rate \( \lambda_k(p) \)
\( C_j \) component arrival renewal process, rate \( \gamma_j \)
\( c_j \) component unit production cost
\( A_k(t) \) cumulative no. of type \( k \) orders assembled up to \( t \)

\( u = (p_u, \gamma_u, A_u) \) admissible policy
( prices, production rates, assembly sequence rule)
\( Q_{u,k}(t) \) order queue-length, \( = O_{u,k}(t) - A_{u,k}(t) \geq 0 \)
\( I_{u,j}(t) \) inventory levels, \( = C_{u,j}(t) - \sum_{k=1}^{K} a_{kj} A_{u,k}(t) \geq 0 \)
Model Formulation - technical assumptions

\( \lambda(p) \) is continuous, differentiable, and the Jacobian matrix is invertible. guarantees \( p(\lambda) \) is unique, continuous, and differentiable.

Customer demand for product \( k \) is strictly decreasing in \( p_k \), but may be increasing in \( p_m, m \neq k \). \( \frac{\partial \lambda_k(p)}{\partial p_k} < 0 \) while \( \frac{\partial \lambda_k(p)}{\partial p_m} \geq 0, m \neq k \).

Increase in the price of one product cannot lead to an increase in the total rate of demand for all products. \( \frac{-\partial \lambda_k}{\partial p_k} > \sum_{m \neq k} \frac{\partial \lambda_m}{\partial p_k} \).

Revenue rates for each product class, \( r_k(\lambda) = \lambda_k p_k(\lambda) \) are concave.

Renewal processes \( O_k \) and \( C_j \) started in steady state at time zero.
Model Formulation - profit expression

infinite horizon discounted profit:

\[
\Pi = \sum_{k=1}^{K} \int_{0}^{\infty} p_k e^{-\delta t} dA_k(t) - \sum_{j=1}^{J} \int_{0}^{\infty} c_j e^{-\delta t} dC_j(t)
\]

\[
\Pi = \sum_{k=1}^{K} \left( \int_{0}^{\infty} p_k e^{-\delta t} dO_k(t) - \int_{0}^{\infty} Q_k(t) e^{-\delta t} dt \right) - \sum_{j=1}^{J} \int_{0}^{\infty} c_j e^{-\delta t} dC_j(t),
\]

where \(Q_k(t)\) is the order queue-length

\[
\int_{0}^{\infty} e^{-\delta t} dO_k(t) - \int_{0}^{\infty} e^{-\delta t} dA_k(t) = \int_{0}^{\infty} \delta e^{-\delta t} Q_k(t) dt
\]
Model Formulation - static planning problem

if we assume that demand and production flow at the long run average rates continuously and deterministically,

\[
\bar{\pi} = \max_{p \geq 0, \gamma \geq 0} \sum_{k=1}^{K} p_k \lambda_k(p) - \sum_{j=1}^{J} \gamma_j c_j
\]

s.t. \[
\sum_{k=1}^{K} a_{k,j} \lambda_k(p) \leq \gamma_j, \quad j = 1, ..., J
\]

– optimal solution \((p^*, \gamma^*)\) assumed to be unique, positive. the first order condition imply that all constraints are tight \((p^*, \gamma^*)\).

– \(\bar{\pi}\) is an upper bound on the expected profit rate.

want to show that under high volume conditions, the optimal prices and production rates are close to \((p^*, \gamma^*)\).
Asymptotic analysis - high demand volume conditions

any strictly increasing sequence \( \{n\} \) in \([0, \infty)\), \( n \) tends to infinity. order arrival rate function \( \lambda^n \), where \( \lambda^k_n(p) = n\lambda_k(p) \), \( k = 1, ..., K \).

\( n\bar{\pi} \) upper bounds the expected profit rate in the \( n^{th} \) system,

\[
\Pi^n \leq \int_0^\infty n\bar{\pi}e^{-\delta t}dt = \delta^{-1}n\bar{\pi}
\]

plug \( (p^*, n\gamma^*) \) into the \( n^{th} \) system, \( n^{-1}\Pi(p^*, n\gamma^*, A^n) \to \delta^{-1}\bar{\pi} \) as \( n \to \infty \), given that \( n^{-1}Q^n \to 0 \) a.s., as \( n \to \infty \).
Asymptotic analysis - proposed assembly policy

component shortage process:

\[ S_j(t) = \sum_{k=1}^{K} a_{kj} O_k(t) - C_j(t) = \sum_{k=1}^{K} a_{kj} Q_k(t) - I_j(t), \quad j = 1, \ldots, J \]

min. instantaneous cost arrangement of queue-lengths and inventory levels \((Q^*(S), I^*(S))\),

\[
\min_{Q,I \geq 0} \sum_{k=1}^{K} P_k^* Q_k
\]

s.t. \[ I_j = \sum_{k=1}^{K} a_{kj} Q_k - S_j \geq 0, \quad j = 1, \ldots, J \]
Asymptotic analysis - proposed assembly policy

for the $n^{th}$ system, the review period $l^n = n^{-\alpha}$, where $\alpha = (4(3 + 2\epsilon_1))^{-1}(6 + 5\epsilon_1) > 1/2$
Asymptotic analysis - system behavior

(See Theorem 1 on page 12 of the Plambeck and Ward paper)
Review on Brownian Motion

A standard Brownian Motion (Wiener process) is a stochastic process $W$ having
1. continuous sample paths
2. stationary independent increments
3. $W(t) \sim N(0, t)$

A stochastic process $X$ is a Brownian motion with drift $\mu$ and variance $\sigma^2$ if

$$X(t) = X(0) + \mu t + \sigma W(t), \quad \forall t$$

then $E[X(t) - X(0)] = \mu t$, $Var[X(t) - X(0)] = \sigma^2 t$.

variance of a Brownian motion increases linearly with the time interval.
Optimality of Nearly Balanced Systems

(See Theorem 2 on page 15 of the Plambeck and Ward paper)
System with delay constraints

propose a near-optimal discrete review control policies, which both sequences customer orders for assembly and expedites component production in an ATO system with delay constraints.