

Guided Study Program in System Dynamics

System Dynamics in Education Project

System Dynamics Group

MIT Sloan School of Management¹

Solutions to Assignment #4
Wednesday, October 21, 1998

Reading Assignment:

Please refer to Road Maps 2: A Guide to Learning System Dynamics (D-4502-4) and read the following papers from Road Maps 2:

- *An Introduction to Feedback, by Leslie A. Martin (D-4691)*
- *Graphical Integration Part One: Exogenous Rates, by Alice Oh (D-4547)*

Also read the following:

- *Study Notes in System Dynamics,² by Michael Goodman: Chapter 2 and Chapter 3, Sections 3.1 to 3.9*

Exercises:

1. An Introduction to Feedback

The purpose of this exercise is to reinforce your understanding of positive and negative feedback, and to prepare you for modeling systems containing both types of feedback.

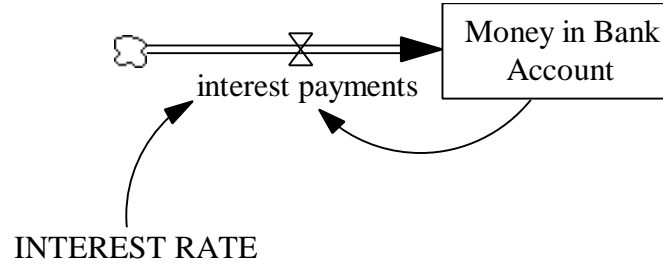
A. The exercises attached to “An Introduction to Feedback” include a simple model of a bank account on page 19. The model exhibits exponential behavior through positive feedback. Using Vensim PLE, build this model and enter initial conditions and constants that you think are realistic. Simulate the model for 50 years. From a graph of model behavior, estimate the doubling time for this system. That is, how many years pass before the amount of money in your bank account doubles? In your assignment solutions document, please include the model diagram, documented equations, and a graph of model behavior.

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² Goodman, Michael R., 1974. *Study Notes in System Dynamics*, Portland, OR: Productivity Press. 388 pp.

D-4716-1

Model diagram:



Model equations:

$\text{interest payments} = \text{Money in Bank Account} * \text{INTEREST RATE}$

Units: dollar/Year

The yearly amount of interest payments.

$\text{INTEREST RATE} = 0.05$

Units: 1/Year

The annual rate of interest.

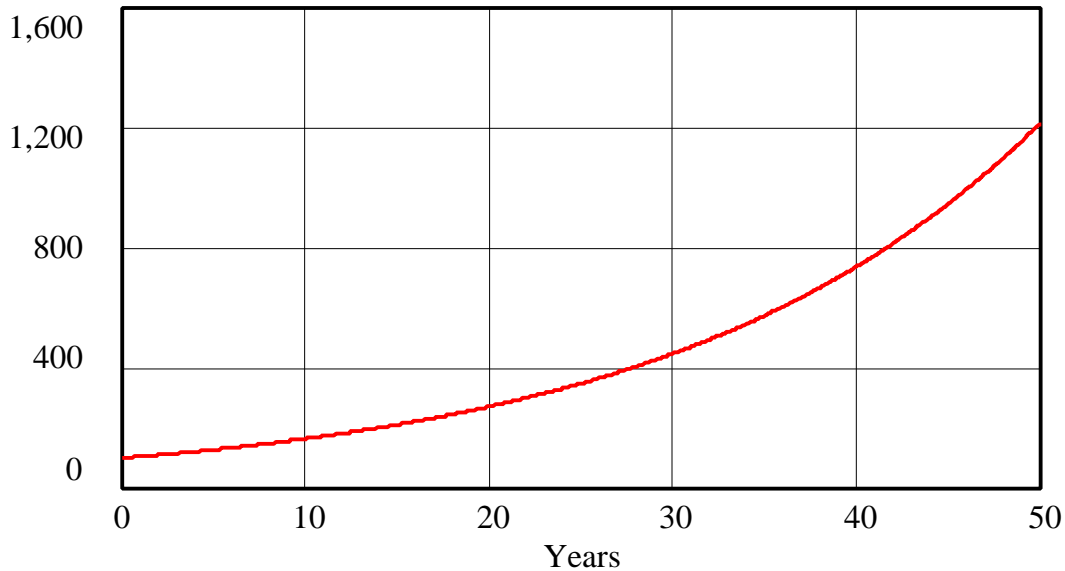
$\text{Money in Bank Account} = \text{INTEG}(\text{interest payments}, 100)$

Units: dollar

The amount of money in the bank account.

Model behavior:

Money in Bank Account



Money in Bank Account : bankA ————— dollar

From the graph of model behavior, one can see that the amount of “Money in Bank Account” doubles approximately after 14 years, if the “INTEREST RATE” is 0.05/year. In fact, the doubling time can be calculated as approximately 0.7 divided by the “INTEREST RATE”:

$$\text{doubling time} = 0.7 / \text{INTEREST RATE}$$

Please note that this formula applies only to simple first-order positive-feedback systems.

B. In real life, you may notice that many systems do not grow unchecked. Unfortunately, our bank accounts are no exception. Let us study, through building another model, how negative feedback prevents us from becoming millionaires (darn!):

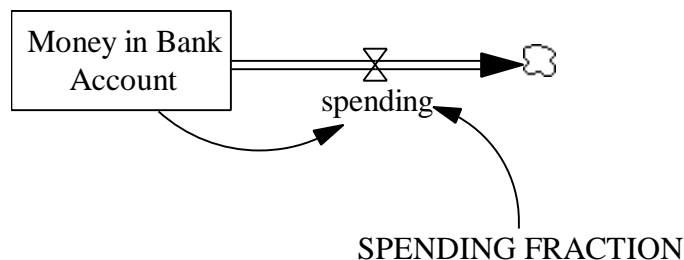
Often, a person’s spending habits depend on the amount of money he currently has in his bank account. Let’s say that you are comfortable with spending about 50% of your current bank balance per year. Build a model that you can use to simulate the behavior in this scenario. If your bank account starts out at \$2000, how long does it take for the balance to halve? If your bank account starts out with \$10,000, how long is it before the balance halves? Again, estimate the half-lives from a graph of the model behavior. How do you intuitively explain the relationship between these two half-lives? In your assignment solutions document, please include the model diagram, documented equations, and a graph of model behavior.

Hint: In order to answer this question correctly, you should make sure to change the time step of the simulation (also known as solution interval or DT). In Vensim PLE, the default setting for the time step is 1 time unit. The time interval denotes how often Vensim PLE solves the equations for all variables. If the time step is set to 1 year, the first time Vensim PLE performs the calculations for the system is at time = 1 year. The model says that each year, 50% of the balance is spent; hence, Vensim PLE removes half the balance at the end of the first year. This would be correct if we assumed that you withdraw the 50% of the balance at one time, at the end of the year. Instead, we assume that the withdrawing is a continuous process.

To make the simulation a more accurate representation of the continuous process, you need to decrease the time step of the simulation. To do this, go to the “Model” menu in Vensim PLE and click on “Time Bounds.” To reduce the value of “TIME STEP,” pull down the menu next to the “TIME STEP” window and choose a value smaller than 1.

As a general rule of thumb, the time step in any simulation should be less than one half, but larger than one-fifth of the smallest time constant in a model; for small models, we recommend using a time step of one-eighth of the smallest time constant. The time constant is the reciprocal of the growth or decay fraction. In this model, the only (and thus smallest) time constant is the reciprocal of the “SPENDING FRACTION,” equal to $1/0.5$, or 2 years. You should therefore choose a time step less than 1 year, say 0.5 years or smaller. When the time step equals 0.5 years, Vensim PLE performs the calculations $1/0.5$ or 2 times a year. Setting the time step to a number greater than one-half of the smallest time constant may generate incorrect system behavior.

Model diagram:



Model equations:

Money in Bank Account = INTEG (-spending, 10000)

Units: dollar

The amount of money in the bank account.

spending = Money in Bank Account * SPENDING FRACTION

Units: dollar/Year

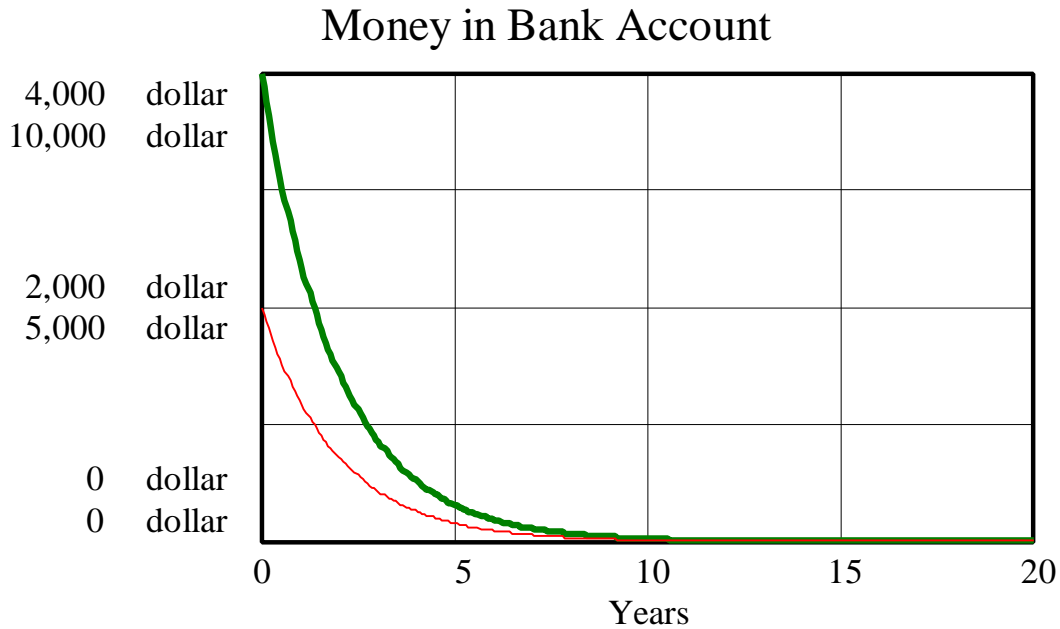
The amount of money spent per year.

SPENDING FRACTION = 0.5

Units: 1/Year

The fraction of current balance that is spent per year.

Model behavior:



Money in Bank Account : bankB2000 ———— dollar
 Money in Bank Account : bankB10000 ———— dollar

From the graph of model behavior, one can see that the time for the “Money in Bank Account” to halve is approximately 1.4 years. In fact, the half-life and the decay constant (the “SPENDING FRACTION” in this example) are related by the formula:

$$\text{half-life} = 0.7 / \text{decay fraction}$$

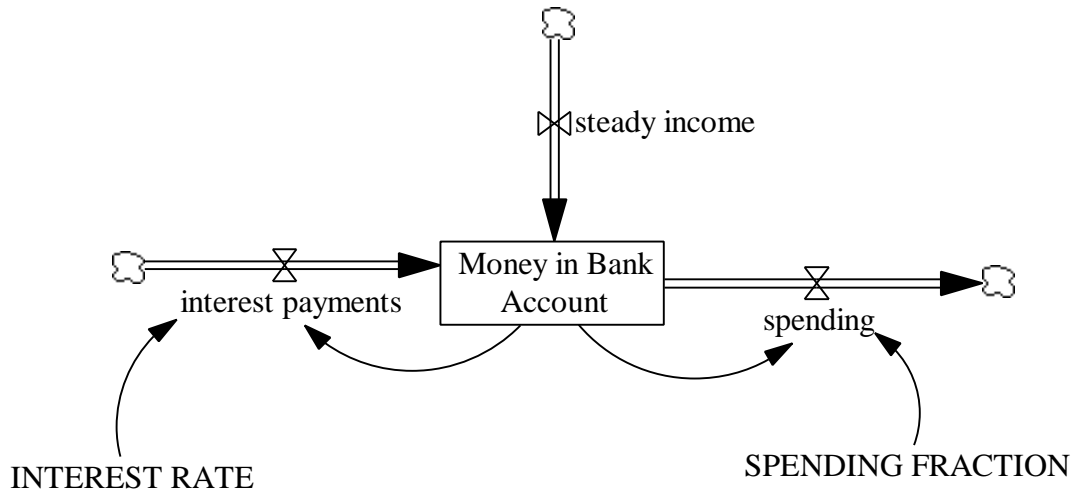
Please note that this formula applies only to simple first-order negative-feedback systems.

No matter what the initial value of the stock is, the half-life is always the same for a given decay fraction. If you start out with more money in your account balance, you spend more money each year. If you start out with less money, you spend less each year. Regardless of the amount of money you have, the fraction that you spend each year is always one-half.

C. Now, take the two models and put them together into one. Start with a stock of \$10,000 and assume that your account earns interest of 5% annually. In addition, each year you have a steady income of \$5000 independent of your account balance. You feel comfortable spending 50% of your account balance per year. Formulate a model of this system and simulate the model. In your assignment solutions document, include the model diagram, documented equations, and a graph of model behavior. How do the

positive and negative feedback loops interact? Do they cancel each other, and if so, when? Does one loop dominate the other?

Model diagram:



Model equations:

$$\text{interest payments} = \text{Money in Bank Account} * \text{INTEREST RATE}$$

Units: dollar/Year

The yearly amount of interest payments.

$$\text{INTEREST RATE} = 0.05$$

Units: 1/Year

The annual rate of interest.

$$\text{Money in Bank Account} = \text{INTEG} (\text{interest payments} + \text{steady income} - \text{spending}, 10000)$$

Units: dollar

The amount of money in the bank account.

$$\text{spending} = \text{Money in Bank Account} * \text{SPENDING FRACTION}$$

Units: dollar/Year

The amount of money spent per year.

$$\text{SPENDING FRACTION} = 0.5$$

Units: 1/Year

The fraction of the current balance spent per year.

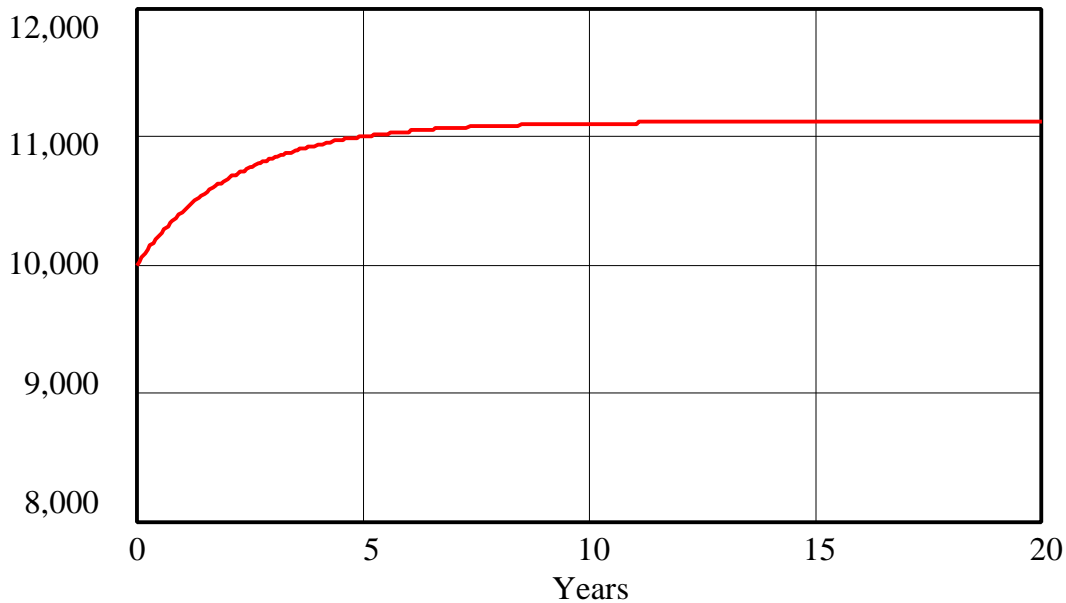
$$\text{steady income} = 5000$$

Units: dollar/Year

The steady income per year.

Model behavior:

Money in Bank Account with steady income of \$5000/year



Money in Bank Account : BankC ————— dollar

The stock of “Money in Bank Account” is approaching an equilibrium value and the behavior is asymptotic. Therefore, the negative-feedback loop dominates the system, driving the system towards equilibrium. The equilibrium value of the stock can be calculated by setting the sum of all inflows equal to the sum of all outflows:

interest payments + steady income = spending

Money in Bank Account * INTEREST RATE + steady income = Money in Bank Account * SPENDING FRACTION

Money in Bank Account * 0.05/year + \$5000/year = Money in Bank Account * 0.5/year

Hence, at equilibrium,

Money in Bank Account = (\$5000/year) / (0.5/year – 0.05/year)

Money in Bank Account = \$11,111.11

Thus, one can find the equilibrium value for any combination of “steady income,” “INTEREST RATE,” and “SPENDING FRACTION.” If the initial amount of “Money in Bank Account” is lower than the equilibrium amount, the negative-feedback loop will cause the value of the stock to increase asymptotically until the equilibrium value is reached. If the initial amount of “Money in Bank Account” is higher than the equilibrium

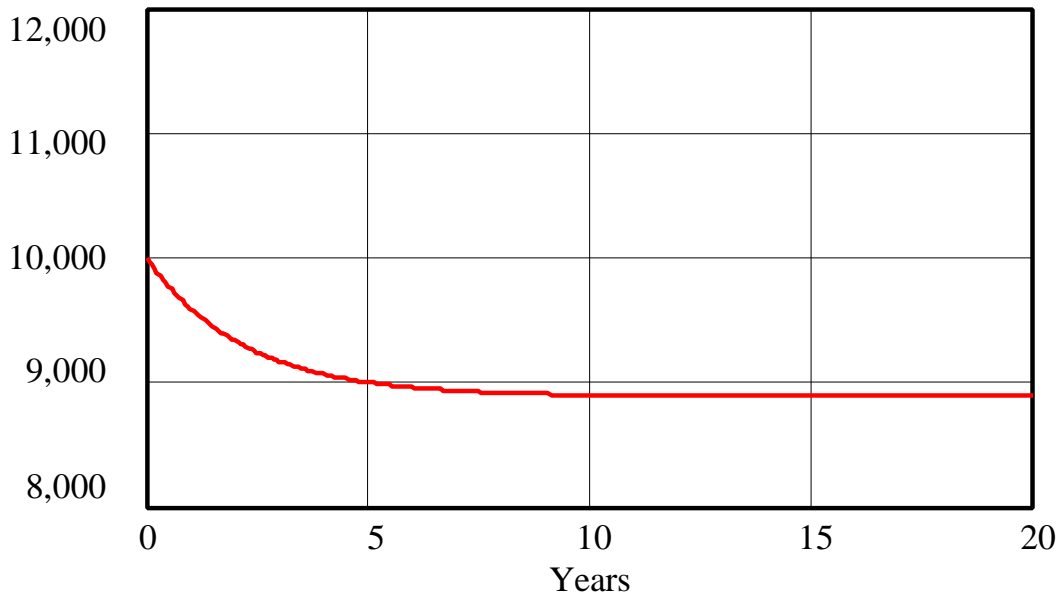
amount, the negative-feedback loop will reduce the value of the stock until the equilibrium value is reached.

One can find out whether a positive or negative-feedback loop is dominant by looking at a graph of the stock behavior. When the positive-feedback loop is dominant, the slope of the graph increases with time, as in the graph in part A. A dominant negative-feedback loop, on the other hand, causes the slope of the graph to decrease, even though the slope may be positive or negative. In this part of the exercise, the slope of the stock graph is positive because the value of the stock is increasing. The slope of the graph, however, is decreasing, that is, becoming less and less positive with time, indicating that the stock is growing more and more slowly as time passes. Eventually, the net flow becomes zero and the stock reaches the equilibrium value.

D. Now, simulate the model developed in part C again with a steady income of \$4000. What happens now and why? Include a graph of model behavior in your assignment solutions document.

With a “steady income” of \$4000/year, the stock exhibits the following behavior:

Money in Bank Account with steady income of \$4000/year

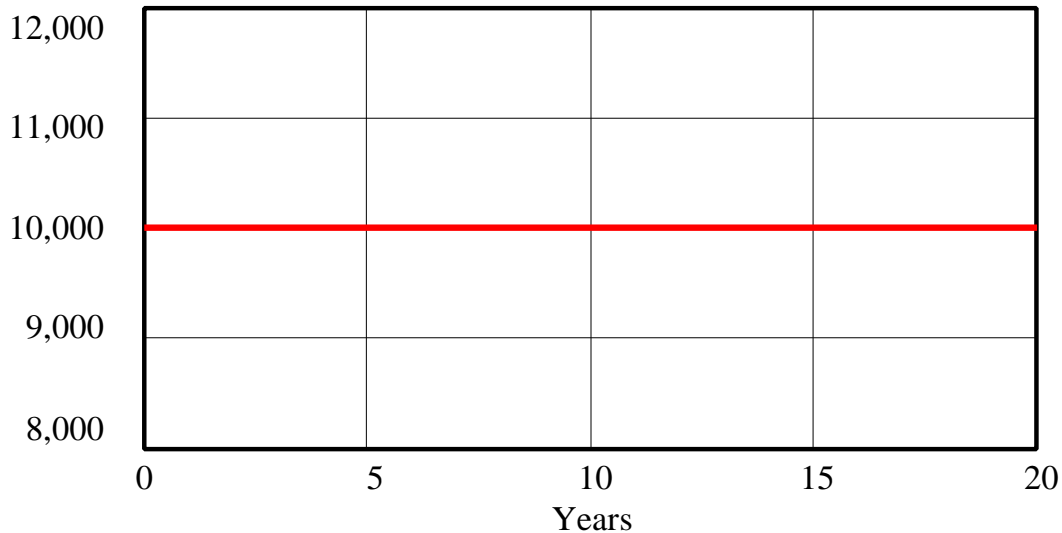


Money in Bank Account : BankD ————— dollar

The system is still dominated by the negative-feedback loop, but the stock is asymptotically decaying instead of growing because the equilibrium value of the stock is lower than the initial amount of “Money in Bank Account.” Using the formula presented in part C, the equilibrium value of “Money in Bank Account” is \$8,888.89.

E. Finally, simulate the model developed in part C with a steady income of \$4500. What happens now and why? Include a graph of model behavior in your assignment solutions document.

Money in Bank Account with steady income of \$4500/year



Money in Bank Account : BankE ————— dollar

The stock remains constant at its initial value because the sum of the inflows is equal to the sum of the outflows. The system is in equilibrium throughout the simulation.

What causes the three stock behaviors in parts C, D, and E to be different? To understand why in each case, the stock increased, decreased or stayed constant, we must examine the inflow and the outflow.

- When steady income is \$5000/year, the initial inflow is:

$$\$10000 * 0.05/\text{year} + \$5000/\text{year} = \$5500/\text{year}$$

The outflow is:

$$\$10000 * 0.5/\text{year} = \$5000/\text{year}$$

Because the inflow is greater than the outflow, the stock immediately increases. As “Money in Bank Account” grows, so does the outflow “spending.” Gradually, the outflow catches up with the inflow and the stock approaches equilibrium.

- When the steady income is \$4000, the initial inflow is:

$$\$10000 * 0.05/\text{year} + \$4000/\text{year} = \$4500/\text{year}$$

Hence, the inflow is \$500/year lower than the outflow of \$5000/year. As a result, the stock decreases. The smaller stock gets, the smaller the outflow, until the inflow and outflow are equal and the stock approaches equilibrium.

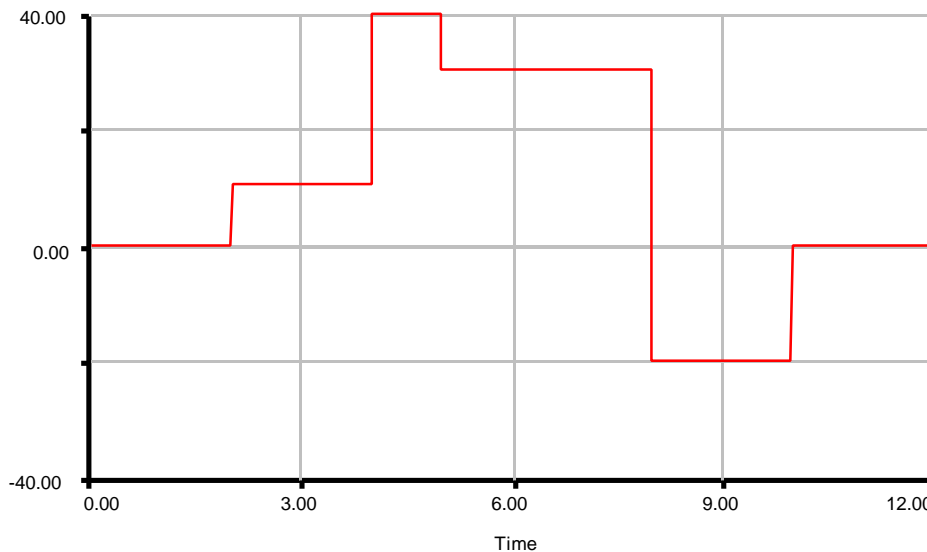
- Finally, when the steady income is \$4500, the initial inflow is:

$$\$10000 * 0.05/\text{year} + \$4500/\text{year} = \$5000/\text{year}$$

Notice that this is exactly equal to the initial outflow. Because the inflow and outflow are equal, the system will not veer from equilibrium.

2. Graphical Integration Exercise Part One: Exogenous Rates

Please graphically integrate the following flow:

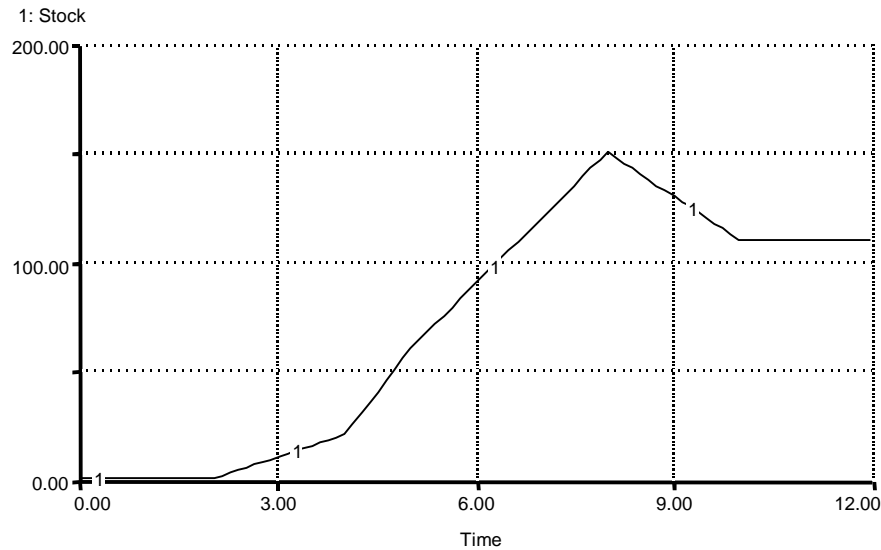


The flow starts out at 0, steps up to 10 at time = 2, then up to 40 at time = 4. At time = 5, the flow drops to 30, then remains steady until dipping to -20 at time = 8. At time = 10, the flow steps back up to 0 and remains constant thereafter.

In your assignment solutions document, you should either submit a graph³ showing the integration of the above flow, or describe the integration verbally. If you choose to describe the stock behavior verbally, you should make sure you clarify what changes take place at a certain time, and what the slope of the stock is at all times.

³ You can submit graphs by creating them in a graphics application and then pasting them into your assignment solutions document, or by faxing them to us at (1-617-258-9405). If neither of these options is convenient, draw the stock by hand, and then describe its behavior carefully in words in your assignment solutions document.

Graphical integration of the graph of the flow should give the following stock graph:



The stock is constant and equal to zero until time = 2.

From time = 2 to time = 4, the flow equals 10, so the stock graph begins to grow with a slope of 10. The change in the value of the stock between time = 2 and time = 4 is:

$$(10 \text{ units of stock / unit of time}) * 2 \text{ units of time} = 20 \text{ units of stock.}$$

Hence at time = 4, the stock reaches the value 20.

From time = 4 to time = 5, the flow equals 40, so the slope of the stock graph is 40. The change in the value of the stock between time = 4 and time = 5 is:

$$(40 \text{ units of stock / unit of time}) * 1 \text{ unit of time} = 40 \text{ units of stock}$$

Hence at time = 5, the stock reaches the value 60.

From time = 5 to time = 8, the flow equals 30, so the slope of the stock decreases accordingly to 30 (note the slight decrease in slope of the graph at time 5). The change in the value of the stock between time = 5 and time = 8 is:

$$(30 \text{ units of stock / unit of time}) * 3 \text{ units of time} = 90 \text{ units of stock}$$

Hence at time = 8, the stock reaches the value 150.

From time = 8 to time = 10, the flow is negative and equals -20 , so the stock starts decreasing with slope -20 . The change in the value of the stock between time = 8 and time = 10 is:

$$(-20 \text{ units of stock / unit of time}) * 2 \text{ units of time} = -40 \text{ units of stock}$$

Hence by time = 10, the stock declines to the value 110.

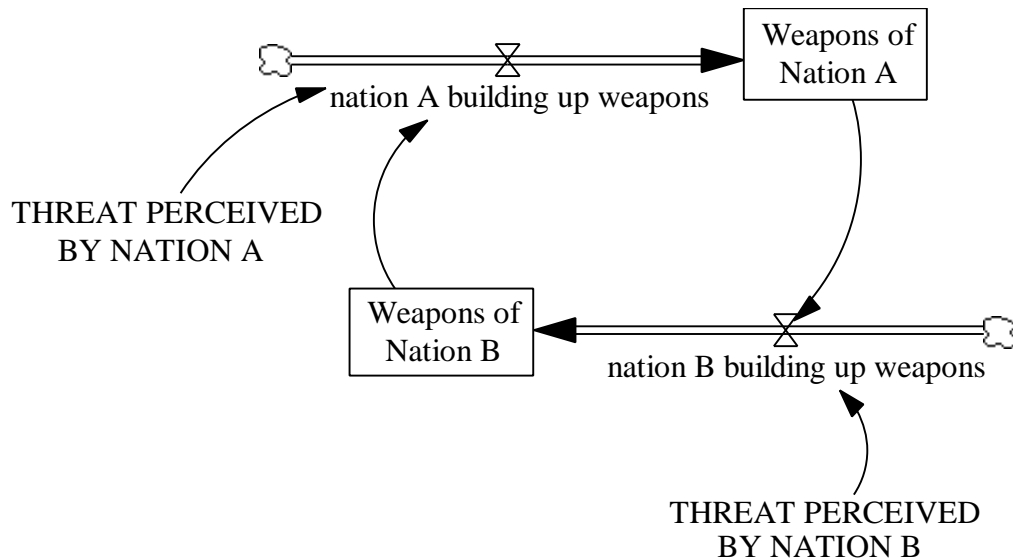
Finally, from time = 10 to time = 12, the flow equals 0, so the slope of the stock graph is 0, and the stock remains constant at 110.

3. Study Notes in System Dynamics, Chapter 2, and Chapter 3, Sections 3.1 to 3.9

A. Read Chapter 2.

B. Chapter 2 (page 16) shows a causal-loop diagram illustrating the feedback loops in arms proliferation of two warring nations. Build a model in Vensim PLE based on the diagram, with initial weapons of each nation equal to 10 missileheads. Make the “threat perceived by nation A” and the “threat perceived by nation B” equal at 0.2/year. Simulate the model and examine the accumulation of weapons for each nation over time. What are the doubling times of the stocks? In your assignment solutions document, include the model diagram, documented equations, and graphs of model behavior.

Model diagram:



Model equations:

$$\text{nation A building up weapons} = \text{Weapons of Nation B} * \text{THREAT PERCEIVED BY NATION A}$$

Units: weapon/Year

The number of weapons Nation A builds each year.

nation B building up weapons = Weapons of Nation A * THREAT PERCEIVED BY NATION B

Units: weapon/Year

The number of weapons Nation B builds each year.

THREAT PERCEIVED BY NATION A = 0.2

Units: 1/Year

The effect of Nation B's weapons on the number of weapons Nation A builds per year.

THREAT PERCEIVED BY NATION B = 0.2

Units: 1/Year

The effect of Nation A's weapons on the number of weapons Nation B builds per year.

Weapons of Nation A = INTEG (nation A building up weapons, 10)

Units: weapon

The total number of weapons accumulated by Nation A.

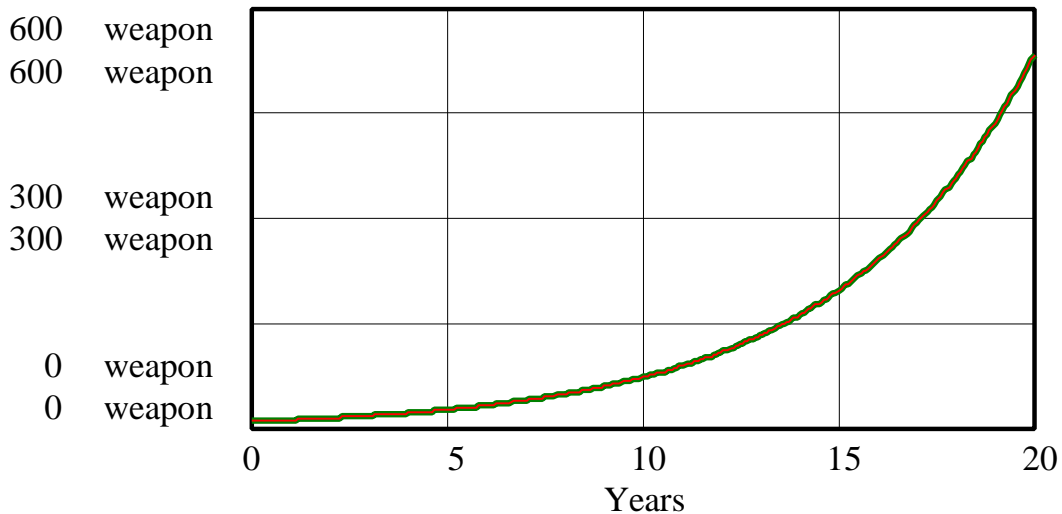
Weapons of Nation B = INTEG (nation B building up weapons, 10)

Units: weapon

The total number of weapons accumulated by Nation B.

Model behavior:

Weapons of Nations A and B - part B



Weapons of Nation A : weapons part B ———— weapon
Weapons of Nation B : weapons part B ———— weapon

Notice that the doubling time of both stocks is approximately $0.7 / 0.2 = 3.5$ years because the time constant for the two flows is the same. The period of simulation, 20 years, spans the doubling time almost 6 times.

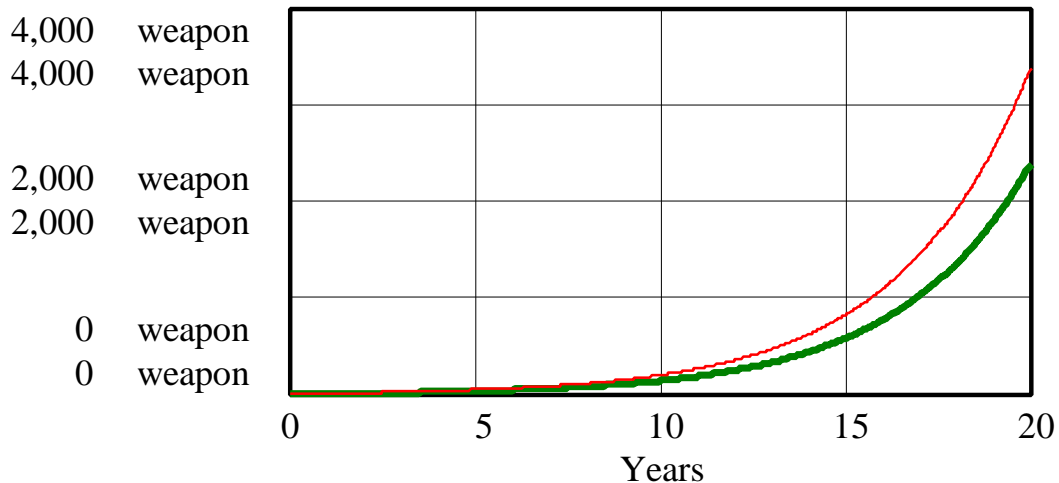
Examining the model structure reveals a positive-feedback loop between the two stocks. Even though the flow to each stock does not directly depend on the current value of that stock itself, each flow depends on the current value of the other stock. The feedback loop contains more than one stock, but it is still a feedback loop: the more weapons nation A has, the greater the threat felt by the nation B, so the more weapons nation B builds up. As nation B accumulates more and more weapons, nation A becomes increasingly threatened and speeds up the build-up of its own weapons. As a result, both nations accumulate weapons exponentially over time, reaching over 500 weapons each after 20 years.

It is important to recognize that the initial conditions and the time constants associated with the stocks are the same. The identical formulation of the two stocks allows them to exhibit the same behavior over time.

C. What if nation A is more sensitive to the threat of armament than nation B? Change the “threat perceived by nation A” to 0.4/year. Simulate the model and discuss the results. How do the numbers of weapons of each nation differ over time? How are the doubling times of the two stocks related? Include graphs of model behavior in your assignment solutions document.

When the “THREAT PERCEIVED BY NATION A” is increased to 0.4/year, the model exhibits the following behavior:

Weapons of Nations A and B - part C



Weapons of Nation A : weapons part C ———— weapon
 Weapons of Nation B : weapons part C ———— weapon

In this case, although the time constant associated with the flow “nation B building up weapons” is unchanged, it is evident that after 20 years, BOTH stocks have reached higher levels than in part B. Nation A ends up with a little over 3000 weapons, while nation B ends up with about 2200 weapons—a significant increase over the previous simulation.

The stock of “Weapons of Nation B” rises faster because its flow depends on the number of “Weapons of Nation A” as well as the “THREAT PERCEIVED BY NATION B.” With a higher “THREAT PERCEIVED BY NATION A,” the stock of “Weapons of Nation A” grows faster than before. Also, because the number of “Weapons of Nation A” directly affects how rapidly nation B builds weapons, this higher constant also speeds up nation B’s weapon build-up, thereby indirectly speeding up nation A’s weapon build-up, and so on, creating positive feedback for both stocks. So the size of Nation B’s stockpile is influenced not only by nation B’s own policy but also by Nation A’s policy.

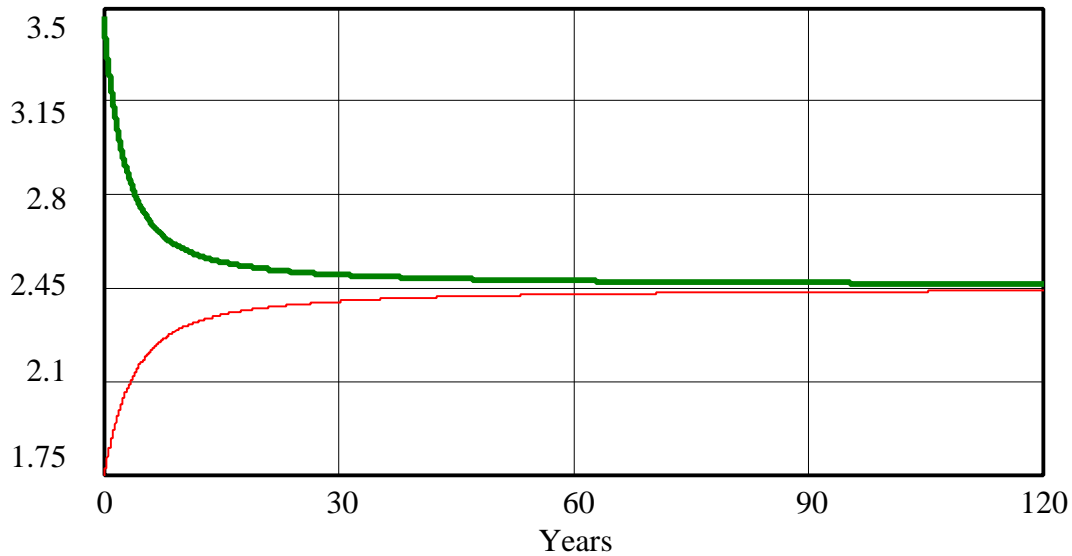
One would expect the doubling time of “Weapons of Nation A” to be twice that of “Weapons of Nation B.” This would be the case if the two stocks grew independently of each other. The growth of each stock, however, affects the growth of the other stock.

Although the growth fraction for the build-up of weapons of each nation is constant (either 0.2 or 0.4 per year), the formula for the doubling time that was used in Exercise 1 cannot be used. The formula

$$\text{doubling time} = 0.7 * \text{time constant} = 0.7 / \text{growth fraction}$$

applies only to first-order systems (systems with only one stock). In a first-order system, the stock affects its own flow. Therefore, the flow changes the stock and the stock changes the flow directly. The arms proliferation system, however, is a second-order system. Because of the dependence of each stock on the other stock, the doubling times of the stocks are not only different but also change with time, as one can observe from graphs of model behavior or from output tables.

Doubling times of the two stocks



Nation A doubling time : doubling time _____ Year
 Nation B doubling time : doubling time _____ Year

As the above graph shows, the initial doubling time of nation A is approximately 1.75 year and the initial doubling time of nation B is approximately 3.5 years. The first-order formula for calculating doubling times can only be used to calculate the initial doubling times, that is, the doubling times at time = 0. Over time, however, the growth fractions of the two stocks interact, causing the doubling time of “Weapons of Nation A” to increase and the doubling time of “Weapons of Nation B” to decrease. The two doubling times converge towards the same value of approximately 2.45 years.

The doubling times of the two stocks are related because the flow into each nation’s stock depends on the level of the other nation’s stock. So even though the time constants are constant (by definition, the time constant, being a constant, cannot depend on anything), the doubling times change with time. When the two doubling times converge, the number of weapons in each nation increases in such a way that the ratio of weapons of nation A to the weapons of nation B is constant at approximately 1.4.

D. Read Chapter 3 up to section 3.9. The examples from chapter 3 will be covered in a later assignment.