Chapter 7

Chapter II: Meaning and Form in Mathematics

But what does it all <u>*mean*</u> *Basel? – Austin Powers*

7.1 Abstract

A new formal system (the pq- system) is presented, even simpler than the MIU-system of Chapter I. Apparently meaningless at first, its symbols are suddenly revealed to possess meaning by virtue of the form of the theorems they appear in. This revelation is the first important insight into meaning: its deep connection to isomorphism. Various issues related to meaning are then discussed, such as truth, proof, symbol manipulation, and the elusive concept, "form". *GEB pp. viii*

7.2 The pq-System

DEFINITION: `xp-qx-` is an axiom, whenever x is composed of hyphens only.
RULE: Suppose x,y, and z all stand for particular strings containing only hyphens. And suppose that `xpyqz` is known to be a theorem. Then `xpy-qz-` is a theorem.
Hofstadter recommeds we try to develop a decision procedure – a test that when applied to any theorem of the `pq-`, either verifies it is a theorem or shows that it is not.

7.3 The Decision Procedure

1. Test first that the theorem is a "well-formed formula."
2. Count the number of hyphens in the first two hyphen-groups.
3. Count the number of hyphens in the third hyphen-group.
4. If the numbers in two and three are equal, you have a theorem!

7.4 Bottom-up *vs.* Top-down

One starts from a "trunk of axioms" (the bottom) and derives up to all the "theorem-leaves", the other takes a leaf of the "theorem-tree" and then tries to follow the branches all the way back to the trunk.

7.5 Isomorphisms Induce Meaning

Recall the definition of an isomorphism from the *Tools for Thinking* chapter. Let us now write down the following definition:

- An **interpretation** is a symbol-word correspondence.

How does interpretation work? Do you really think that the `pq-` system *gains* meaning because of the coincidence with addition rules? Where does meaning exist?

7.6 Active vs. Passive Meanings

Hofstadter tells us the most important thing to get out of the chapter (p. 51):

> The pq-system seems to force us into recognizing that *symbols of a formal system, though initially without meaning, cannot avoid taking on "meaning" of sorts, at least if an isomorphism is found.*

7.7 Meaningless and Meaningful Interpretations

Is meaning subjective?

7.8 Double-Entendre!

The role of intuition in discovering "truths" must then be accompanied by the formality of string manipulation. Where does this break down? Does intution lead us astray or is it essential?

7.9 Formal Systems and Reality

What do you think of the following excerpt? (GEB pp. 53-54)

> Can all of reality be turned into a formal system? In a very broad sense, the answer might appear to be yes. One could suggest, for instance, that reality is itself nothing but one very complicated formal system. Its symbold do not move around on paper, but rather in a three-dimensional vacuum (space); they are the elementary particles of which everything is composed. (Tacit assumption: that there is an end to the descending chain of matter, so that the expression"elementary particles" makes sense.) The "typographical rules" are the laws of physics, which tell how, given the positions and velocities of all particles at a given instant, to modify them, resulting in a new set of positions and velocities belonging to the "next" instant. So the theorems of this

grand formal system are the possible configurations of particles at different times in the history of the universe. The sole axiom is (or perhaps, *was*) the original configuration of all the particles at the "beginning of time". This is so grandiose a conception, however, that it has only the most theoretical interest; and besides, quantum mechanics (and other parts of physics) cast at least some doubt on even the theoretical worth of this idea. Basically, we are asking if the universe operates deterministically, which is an open question.

7.10 Mathematics and Symbol Manipulation

How do we prove that isomorphisms with the real world are perfect?

7.11 The Basic Laws of Arithmetic

In what physical contexts are basic truths like

$$1 + 1 = 2$$

in fact false? Consider water drops merging and splitting, clouds, and other continuous physical situations. How do we know that the "math-reality" isomorphism holds?

7.12 Ideal Numbers

If numbers seem so fundamental, are they in some respects *more real* than things in reality, such as atoms and birds. Consider Descarte's skepticism of the senses, and consider the philosophical school of Platonism.

7.13 Euclid's Proof

Once again, consider which do you more strongly believe:

- There are infinitely many primes
- The Sky is blue

How should our *belief system* reflect this choice?

7.14 Getting Around Infinity

Abstraction as a tool for thinking.

7.15 Study Questions

Let's get a little more specific here.

7.15.1 The pq- System

1. What is the complete list of allowed symbols in the pq-system? Be sure to take a look at the Rule on p.47. Should your answer include "x", "y", and "z"?

2. What makes an axiom (as defined on p.47) an axiom? Write down examples of some strings that are axioms and some that are not.

3. How many axioms are there in the pq-system?

4. Suppose that - - p - - q - - is a theorem. What theorem can you immediately deduce from it?

5. Hofstadter suggests that you fool around with the pq-system, looking for a way of perceiving at a glance whether or not a string is a theorem. Good idea! Why not give it a try?

7.15.2 The Decision Procedure; Bottom-up versus Top-down

1. How would you represent in symbols Hofstadter's statement on p.47 that "every theorem of the pq-system has three separate groups of hyphens"?

2. Do you think you could <u>prove</u> that statement, using the definition of axioms and the one rule of production?

3. Can you provide a short series of steps by which you could determine whether or not a string is an axiom in the pq-system?

4. In the last paragraph in this section (on p.48), Hofstadter provides a general description of a procedure to progressively shorten a string to test whether it is or is not a theorem. Apply that procedure to the following string to determine whether or not it is a theorem:

   - - - - p - - - - - - q - - - - - - -

5. Follow the rules listed at the top of p.49. What do you make of Hofstadter's statement that this procedure "...can't fail to produce every theorem of the pq-system..."?

7.15.3 Isomorphisms Induce Meaning; Meaningless and Meaningful Interpretations

1. Is the pq-system equivalent to addition? Why or why not?

2. Maps are examples of documents in which symbols are given meaning through an isomorphism. Give examples of symbols used in maps and their interpretations. Note that this "language" isn't very complex: it would be unusual to make a sentence out of symbols from a map (but give it a try!). Think of other instances where symbols are given meaning via an isomorphism.

3. How do we know which interpretation to apply? What's wrong with p $\leftrightarrow$ horse?

7.15.4 Active versus Passive Meanings and Double-Entendre

1. What is the "Requirement of Formality" Hofstadter warns us about at the top of p.52? Do something with respect to the pq-System that would cause him to say, "See? I told you so!"

2. Try out the alternative interpretation of the symbols of the pq-System suggested at the bottom of p.52. How can you be sure that all the old theorems are still true?

3. What is the difference between meaning in a formal system and meaning in a human language?

4. What is the difference between active and passive meaning?

7.16 Study Questions

More specifically:

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