Permutations, Combinations, Partitions

Vina Nguyen
HSSP – July 20, 2008
Review of last class

- What is Bayes’ rule?
Review of last class

- What is the total probability theorem?
What does “A is independent from B” mean?
Review of last class

- How do we test for independence?
If we have probability, and *conditional probability*...

We can have independence, and *conditional independence* too.
Definition:
\[ P(A \cap B \mid C) = P(A \mid C)P(B \mid C) \]
given C, A and B are independent

Another way to write this:
\[ P(A \mid B \cap C) = P(A \mid C) \]
Example: Biased Coin Toss

- We have two coins: blue and red
- We choose one of the coins at random (probability = $\frac{1}{2}$), and toss it twice
- Tosses are independent from each other given a coin
- The blue coin lands a head 99% of the time
- The red coin lands a head 1% of the time

Events: $H_1 = 1^{st}$ toss is a head
        $H_2 = 2^{nd}$ toss is a head
Example: Biased Coin Toss

- Tosses are independent from each other GIVEN the choice of coin
Problem #4: Biased Coin Toss

- What if you don’t know what coin it is? Are the tosses still independent?
Bayes’ rule
Independence
Conditional Independence

Things are not always what they seem! But with these tools you can calculate the probabilities accurately
Counting in Probability

- Where have we seen this?
  - When sample space is finite and made up of equally likely outcomes
  - \( P(A) = \frac{\# \text{ elements in } A}{\# \text{ elements in } \Omega} \)

- But counting can be more challenging...
Use the tree to visualize stages
Stage 1 has $n_1$ possible choices, stage 2 has $n_2$ possible choices, etc...
Divide & Conquer

- All branches of the tree must have the same number of choices for the same stage
The Counting Principle

- An experiment with m stages has

\[ n_1 n_2 \ldots n_m \] results,

where

- \( n_1 = \# \) choices in the 1\textsuperscript{st} stage,
- \( n_2 = \# \) choices in the 2\textsuperscript{nd} stage,
- \( \ldots \)
- \( n_m = \# \) choices in the m\textsuperscript{th} stage
$k$-permutations

- How many ways can we pick $k$ objects out of $n$ distinct objects and arrange them in a sequence?
- Restriction: $k \leq n$
Example: M&M’s

- Pick 4 colors of M&Ms to be your universal set
- How many 2-color sequences can you make?
At each stage, how many possible choices are there? [Use the counting principle]
Formula for $k$-permutations

- Start with $n$ distinct objects
- Arrange $k$ of these objects into a sequence

$\#$ of possible sequences:

$$= \frac{n!}{(n - k)!}$$
Special case: $k=n$

- Formula reduces to: $n!$

- This makes sense – at every stage we lose a choice: $(n)(n-1)(n-2)\ldots(1)$
Combinations

- Start with $n$ distinct objects
- Pick $k$ to form a set

How is this different from permutations?
- Order does NOT matter
- Forming a subset, not a sequence
Example: M&M’s

- Pick 4 colors as the universal set
- How many 2-color combinations can you create?

Remember that for combinations,

\[
\{ \text{green}, \text{red} \} = \{ \text{red}, \text{green} \}
\]
Deriving a formula

- Permutations =
  - 1. Selecting a combination of \( k \) items
  - 2. Ordering the items

- How many ways can you order a combination of \( k \) items?
Deriving a formula

(# k-permutations) =

(# ways to order k elements) \times (# of combinations of size k)
Formula for combinations

- Start with $n$ distinct objects
- Arrange $k$ of these objects into a set

# of possible combinations:

$$= \frac{n!}{k! (n-k)!}$$
Another way to write combinations

- “n” choose “k”

\[ \binom{n}{k} \]

- Side note: this is also known as the “binomial coefficient,” used for polynomial expansion of the binomial power [outside of class scope]
Partitions

- We have a set with \( n \) elements
- Partition of this set has \( r \) subsets
- The \( i \)th subset has \( n_i \) elements

- How many ways can we form these subsets from the \( n \) elements?
Example: M&Ms

- 6 total M&Ms
  - 1 of one color
  - 2 of one color
  - 3 of one color

- How many ways can you arrange them in a sequence?
Example: M&M’s

- One perspective
  - 6 slots = 3 subsets (size 1, size 2, size 3)
  - Each subset corresponds to a color
- At each stage, we calculate the number of ways to form each subset
Example: M&M’s

- Stage #1: Place the first color
- 6 possible slots
- Need to fill 1 slot

# combinations: \( \binom{6}{1} \)
Example: M&M’s

- Stage #2: Place the second color
- 5 possible slots
- Need to fill 2 slots

# combinations: \( \binom{5}{2} \)

Notice how it does not matter which M&M we place in which slot – this implies order does not matter \( \rightarrow \) use combinations
Example: M&M’s

- Stage #3: Place the third color
- 3 possible slots
- Need to fill 3 slots

# combinations: \( \binom{3}{3} \)
Deriving a formula for partitions

- Solution to our example:
  \[
  \binom{6}{1} \binom{5}{2} \binom{3}{3}
  \]

- Generalized form?
Start with $n$-element set (no order)
In this set, there are $r$ disjoint subsets
The $i$th subset contains $n_i$ elements

How many ways can we form the subsets?

$$\frac{n!}{n_1!n_2!...n_r!}$$
Problem Revisited

- A class has 4 boys and 12 girls. They are randomly divided into 4 groups of 4. What’s the probability that each group has 1 boy?

- Use counting methods (partitions) this time
Summary

- The Counting Principle
- Permutations
- Combinations
- Partitions