Lecture 21 Blackboard #1

Rigid Body Kinematics
- Real Objects: mass, density, shape
- Motion: Translation & CM Rotation about axes
- Inertial Frame

Point P moves in a circle
\[ \dot{r} = \omega \times r \]
\[ \dot{\omega} = \frac{1}{I} \mathbf{M} \]
\[ \mathbf{r} = \mathbf{R} \theta \]
\[ \mathbf{R} = \mathbf{R} \theta \]

Angular Velocity: \( \mathbf{w}(t) \)
\[ \mathbf{w} = \frac{\mathbf{r}_2 - \mathbf{r}_1}{t_{2} - t_{1}} \]
\[ \mathbf{R} = \frac{\mathbf{r}_2 - \mathbf{r}_1}{t_{2} - t_{1}} \]

Angular Acceleration: \( \mathbf{a}(t) \)
\[ \mathbf{a} = \frac{\mathbf{\dot{a}}}{t_{2} - t_{1}} \]

Fixed Rotation Axis:
- Direction: \( \mathbf{z} = \text{d} \mathbf{w}/\text{d}t \)
- Changing Axis: No!!

Note: Every point: same \( \omega(t) \)!!

Suppose \( \omega(t) = \omega_0 \): constant.
Translation = \( 2\pi \) radians
Lecture 21 Blackboard #2

Angular Motion: $\dot{\theta} = \text{constant}$

- Fixed Axis
  - Ignored vector notation (± direction)
  - Also true for axes in linear translation (rotating)

$$\frac{d\theta}{dt} = \theta$$

$$\omega = \omega_0 + \alpha t$$

Period: $T = \frac{2\pi}{w}$

Frequency: $f = \frac{1}{T} = \frac{w}{2\pi}$

Angular Acceleration: $\alpha(t)$

Fused Rotation Axis:

direction $\hat{z} = \text{dir.} \overrightarrow{w}$

Changing Axis: No!!!

Not: Every count: same $w(t)$!!!

Angular $\rightarrow$ Linear Velocity and Acceleration

Every particle moves in a curve about axis $\theta$

$$s = R \phi$$

Angular $= \frac{dR}{dt} \phi + R \omega$$

$$\omega = \frac{dR}{dt} + \frac{R}{R} \phi$$

$$\alpha = \frac{dR}{dt} + \frac{R}{R} \phi = \frac{R}{R} \phi$$

$$\alpha = \frac{dR}{dt} + \frac{R}{R} \phi$$

$r$, $r'$, $r''$: Prop to $R$!!

If $\alpha = 0$, $\alpha_t = 0$, $\alpha_r = 0$
Lecture 21 Blackboard #3

Rotation Kinetic Energy

\[ K = \frac{1}{2} I \omega^2 \]

- Moment of Inertia
- Depends on axis of rotation
- Particles at large R contribute most!

\[ I = \frac{1}{2} m R^2 \]

\[ a, M \quad \rightarrow \quad \text{Riemann to Linear Motion} \]
\[ a, I \quad \rightarrow \quad \text{Riemann to Rotational Motion} \]

Example 4 Rotating Masses

- \( \omega = \omega_0 \), any velocity constant
- \( \alpha = 0 \)

1) Rotate about \( z \)-axis:

\[ I_z = \sum m_i r_i^2 = Ma^2 + \frac{1}{2} m_2 a^2 + \frac{1}{2} m_3 a^2 \]

\[ K_z = \frac{1}{2} I_z \omega^2 = \frac{1}{2} \left( Ma^2 + \frac{1}{2} m_2 a^2 + \frac{1}{2} m_3 a^2 \right) \]

2) \( K_y > K_y \)

3) \( I_y > I_y \)

\[ k_y > k_y \]
Examples: 4 Rotating Masses

\[ \omega = \omega_0 \text{ any related constant} \]
\[ a = \dot{\omega} = 0 \]

a) Rotate about the y-axis:
\[ I_y = \sum m_i r_i^2 = m_1^2a_1 + m_2^2a_2 + 0 = 2m_2a \]
\[ K_y = \frac{1}{2} I_y \omega_y^2 = \frac{1}{2} \left( 2m_2a \right) \omega_y^2 \]
\[ I_y > I_z \]
\[ k_z > k_y \]

Rigid Bodies - Moments of Inertia:

\[ k = \frac{1}{2} I \omega_0^2 \text{ KE for rotation about fixed axis.} \]
\[ \text{dm} = \text{element of mass}. \]
\[ I = \sum m_i^2 \text{dm} \text{ about not fixed axis.} \]
\[ I = \mu^2 \sum \text{dm} = \int r^2 \text{d}m. \]

Let \( s(x,y,z) = \) local density (mass/volume)

\[ I = \frac{M}{V} \frac{1}{2} \int r^2 \text{d}V \]

where \( s = \frac{M}{V} \)}
Example: Moments of Inertia.

- Thin Rod: $I = \frac{1}{12} ML^2$
- Solid Disk: $I = \frac{1}{2} MR^2$
- Ring: $I = MR^2$
- Solid Sphere: $I = \frac{2}{5} MR^2$
- Hollow Sphere/Shell: $I = \frac{2}{3} MR^2$
- Plate: $I = M (\frac{1}{2} L^2 + \frac{1}{12} a^2)$

Example: Uniform Hollow Cylinder

- Outer Radius: $R_o$
- Inner Radius: $R_i$
- Height along center: $L$
- Mass of cylindrical shell:
  - Volume: $V = 2\pi rL (dr)$
  - Mass: $m = \rho V = 2\pi \rho rL (dr)$
- Mass of cylinder:
  - Mass: $M = \pi LR_o^2 R_0$
  - $I = \frac{1}{2} \pi R_o^4 L$
Example: Uniform-Hollow Cylinder

Outer Radius: \( R_2 \)
Inner Radius: \( R_1 \)

Volume of Shell: \( V = 2\pi R L (dV) \)
Mass of Shell: \( dm = \rho dV \)

Mass of Cylinder: \( M = \pi L \rho (R_2^2 - R_1^2) \)

Moment of Inertia:

- If cylinder is solid: \( R = 0 \)
  \[ I = \frac{1}{2} MR^2 \]
- If cylinder is a thin shell: \( R = R_2 - R_1 \)
  \[ I = MR^2 \]

Same as for a mass \( M \) located at a radius \( R \)

Uniform Thin Rod:

Length: \( L \)

Mass: \( M \)

Slice \( dx \) has mass \( dm \)

\[ dm = M \left( \frac{dx}{L} \right) \]

Moment of Inertia:

- With respect to \( x \)-axis:
  \[ I_x = \int_{-L/2}^{L/2} \rho x^2 dm = \frac{M L^2}{12} \]