Example

Particle on a string at radius $r_1$ is moving in a circle with angular velocity $\omega_1$.

String shortened to new radius $r_2$.

What is new angular velocity $\omega_2$?

Ans: shortening string requires a radial force only on particle — no torques.

\[ I_i \omega_i = I_f \omega_f \]
\[ m r_1^2 \omega_i = m r_2^2 \omega_2 \]
\[ \omega_2 = \left( \frac{r_1}{r_2} \right)^2 \omega_1 \]

Conservation of angular momentum about a fixed axis.

Ex: Skaters, gymnasts, ballet dancers change ang. velocity by altering moment-of-inertia by extending arms etc.
Example: Spinning Bicycle Wheel

Student holds a spinning bicycle wheel. Initially student and stool at rest. Wheel spins in horizontal plane with initial ang. mom. \( \vec{L}_0 \) up.

Wheel is inverted through center by 180°. What happens?

System = student + wheel + stool.
Total ang. mom = \( \vec{L}_0 \) [Wheel only spinning]

As wheel is inverted, student supplies a torque—it is internal to the system. No external torques acting on the system about vertical axis. Free spinning wheel.

\( \therefore \) Total angular momentum is conserved.

Initial:\n\( \vec{L}_{\text{system}} = \vec{L}_0 \)

Finally:\n\( \vec{L}_{\text{system}} = \vec{L}_{\text{student+stool}} + \vec{L}_{\text{wheel}} \)

\( \therefore \vec{L}_0 = \vec{L}_{\text{student+stool}} + \vec{L}_{\text{wheel}} \)

\( \vec{L}_0 = \vec{L}_{\text{st+stool}} - \vec{L}_0 \)

\( \therefore \vec{L}_{\text{st+stool}} = 2\vec{L}_0 \)
If $I_p$ = moment of inertia (student + stool) \\
\[ I_p \omega_p = 2I_0 \omega_0 \]

If wheel is held at an angle $\Theta$ measured from the vertical axis, the student acquires an ang. mom. given by \\
\[ l_0 (1 - \cos \Theta) \]

In this case the ang. mom. precesses about axis of rotation of platform. A horizontal torque supplied by the student then accounts for this precession.
Work and Energy in Rotational Motion

A force applied to a moving body does work on the body. The work done by the force $\overrightarrow{F}$ as the body rotates through a small distance $ds = r\,d\theta$ in a time $dt$ is

$$dW = \overrightarrow{F} \cdot d\overrightarrow{s} = (F \sin \phi) r \, d\theta$$

$F \sin \phi =$ tangential component of $\overrightarrow{F}$. Radial component does no work since it is perpendicular to $d\overrightarrow{s}$.

$$(F \sin \phi) r = \tau_z$$

(Torque about axis of rotation)

$$dW = \tau_z \, d\theta$$

$$W = \int \tau_z \, d\theta = \tau_z \Delta \theta$$ for constant torque

Power

$$P = \frac{dW}{dt} = \tau_z \frac{d\theta}{dt} = \tau_z \omega$$
Work-Energy Theorem in Rotational Motion

The work done by the torque changes the KE of the body.

\[ \tau \rightarrow \alpha \quad \omega_1 \rightarrow \omega_2 \]

\[ \tau = I \alpha = I \frac{d\omega}{dt} = I \frac{d\omega}{d\theta} \frac{d\theta}{dt} = I \frac{d\omega}{d\theta} \theta \]

\[ \tau d\theta = dW = I\omega d\omega \]

Total work done is

\[ W = \int_{\theta_1}^{\theta_2} \tau d\theta' = \int_{\omega_1}^{\omega_2} I\omega' d\omega' = \frac{1}{2} I \omega_2^2 - \frac{1}{2} I \omega_1^2 \]

Angular velocity: \( \omega_1 \rightarrow \omega_2 \)

Angular displacement: \( \theta_1 \rightarrow \theta_2 \)

\[ W = AK = K_f - K_i = \frac{1}{2} I \omega_2^2 - \frac{1}{2} I \omega_1^2 \]

If the force acting is conservative (gravity/spring), then the work done is the negative of the change in PE.

\[ -U_a + U_i = \frac{1}{2} I \omega_2^2 - \frac{1}{2} I \omega_1^2 \]

or \[ \frac{1}{2} I \omega_1^2 + U_i = \frac{1}{2} I \omega_2^2 + U_a \]

\[ \Rightarrow \text{Cons. of Mechanical Energy in Rot. Motion} \]

\[ E = \frac{1}{2} I \omega^2 + U = \text{Constant} \]
Example: Rotating Rod

A uniform rod of length \( L \) and mass \( M \) rotates about its end. What is angular velocity at its lowest position?

Use Energy Conservation. Assume zero of PE is at midpoint when rod is hanging down.

\[
E_i = E_f
\]

\[
U_i + \frac{1}{2}I\omega_i^2 = U_f + \frac{1}{2}I\omega_f^2
\]

\[
\frac{1}{2}mgL + 0 = 0 + \frac{1}{2}\left(\frac{1}{3}I\omega^2\right)
\]

\[
\omega = \sqrt{\frac{3g}{L}}
\]

\( L = 1 \text{ m} \)

\( \omega = 5.42 \text{ rad/s} \)

What is linear velocity?

Center: \( v_c = \omega R = \frac{L}{2} \omega = \frac{1}{2} \sqrt{3gL} \)

End: \( v_e = \omega R = L \omega = \sqrt{3gL} \)
Example: Connected Masses

Two masses connected by string passing over a pulley of moment-of-inertia $I$. What are linear velocities of masses after they move a height $h$?

No friction: Total energy is conserved.

$E_1 = E_2$

$k_1 + u_1 = k_2 + u_2$

$0 = \frac{1}{2} m_1 v_1^2 + \frac{1}{2} m_2 v_2^2 + \frac{1}{2} I \omega^2 + m_1 g h - m_2 g h$

$v = R \omega$

$\frac{1}{2} I \omega^2

Solve for $v = \left[ \frac{2 (m_2 - m_1) g h}{m_1 + m_2 + \frac{I}{R^2}} \right]^{1/2}$

Alternate: $\omega = I \omega$, solve for $d$ and subsequent motion.
**Angular Impulse**

We showed previously that if we defined a linear impulse

\[
\vec{I} = \int_{t_1}^{t_2} \vec{F}(t) \, dt
\]

The change in linear momentum

\[
\frac{d\vec{p}}{dt} = \vec{F}
\]

in a time interval \( \Delta t = t_2 - t_1 \)

\[
\Delta \vec{p} = \vec{p}_f - \vec{p}_i = \vec{I}
\]

\[
\frac{d\vec{r}}{dt} = \vec{c}
\]

\[
\vec{r}_f - \vec{r}_i = \Delta \vec{r} = \vec{c} \Delta t
\]

\[
= \vec{J} \text{ (ang. impulse)}
\]

\[
\vec{J} = \int_{t_1}^{t_2} \vec{c}(t) \, dt
\]

Applied torque as a function of time.

Change in \( \vec{r} \) is equal to the angular impulse.
Example

Two disks with moments of inertia $I$ and $I'$ are rotating with velocities $\omega_0$ and $\omega_0'$. Disks pushed together by an external force parallel to axis of rotation. No torques about this axis. Disks rotate with final angular velocity $\omega$.

Disks exert torques on each other. Corresponding impulses

\[ J_\theta = -J'_\theta \]

\[ J_\theta = I\omega - I\omega_0 \quad \text{[Large disk]} \]

\[ J'_\theta = I'\omega - I'\omega'_0 \quad \text{[Small disk]} \]

\[ I\omega - I\omega_0 = (I'\omega - I'\omega'_0) \quad \text{[} J_\theta = -J'_\theta \text{]} \]

\[ \omega = \frac{I\omega_0 + I'\omega'_0}{I + I'} \]

Internal Torques: Total $I^2$ is conserved.

Totally inelastic rotational collision!!
Kinetic Energy is not conserved.
\[ h_i = I\omega_0 + I'\omega_0' \]

\[ h_f = (I + I')\omega \]

\[ h_i = h_f \]

\[ I\omega_0 + I'\omega_0' = (I + I')\omega \]

\[ \omega = \frac{I\omega_0 + I'\omega_0'}{I + I'} \]
Rotation plus translation

Generalize idea of fixed axis rotation

i) Axis of rotation is fixed in an inertial frame

or

ii) Axis of rotation passes through the CM of the body and maintains a constant direction in space.

- in each case every point in body has a linear speed $v=r\omega$ due to rotation.
- in case ii) additional component due to translation of CM.
- rolling is a special motion of case ii)
Rolling Cylinders/Spheres

A uniform cylinder of radius $R$ rolling on a flat surface. Cylinder rotates through angle $\theta$, its cm moves a distance $s = R\theta$.

The velocity and acceleration of the cm for pure rolling motion (no slipping) are given by:

\[
\dot{v}_c = \frac{ds}{dt} = R\dot{\theta} = Rw \\
\ddot{a}_c = \frac{d\dot{v}_c}{dt} = R\ddot{w} = R\alpha
\]

Different points on the object have different linear velocities. The linear velocity of any point is in a direction perpendicular to the line from that point to the point of contact $P$.

The contact point is at rest relative to the surface. An axis through $P$ and perpendicular to $\dot{v}_c$ is the instantaneous axis of rotation.

All points on a rolling body move in a direction perpendicular to an axis through the contact point $P$. The center of mass moves with a velocity $\dot{v}_c$, while the point $P'$ moves with the velocity $2\dot{v}_c$.
A general point \( P \) on the cylinder has both horizontal and vertical components of velocity. Points \( P \) and \( P' \) are of special interest.

\[
P: \quad v_P = 0
\]
\[
P': \quad v_{P'} = 2R\omega
\]

\[
\begin{align*}
\text{Translation} & \\
& \cdot \text{All points have the same speed } v = \nu_c
\end{align*}
\]

\[
\begin{align*}
\text{Rotation} & \\
\omega R &= \nu_c \\
& \cdot \text{All points have the same ang. velocity.} \\
& \cdot \text{Speed prop. to distance from axis.}
\end{align*}
\]

\[
\begin{align*}
\text{Translation + Rotation} & \\
v_P &= \nu_c - \omega R = 0 \\
v_c &= \nu_c + 0 = \nu_c \\
v_{P'} &= \nu_c + \omega R = 2\nu_c
\end{align*}
\]
The combined effects of the translation of the CM and rotation about an axis through the CM are equivalent to a pure rotation with the same angular speed about an axis through the point of contact of the rolling body.

**Kinetic Energy**

Total KE is given by:

$$K = \frac{1}{2} I_p \omega^2$$

$$I_p = I_c + mR^2 \quad \text{[Parallel Axis Thm]}$$

$$\therefore \quad K = \frac{1}{2} I_c \omega^2 + \frac{1}{2} mR^2 \omega^2$$

$$= \frac{1}{2} I_c \omega^2 + \frac{1}{2} mV_c^2 \quad \text{[consistent]}$$

Total KE for an object undergoing pure rolling motion is the sum of the rotational KE about the CM plus the translational KE of the CM.
Translation and Rotation

Theorem: Displacement of a rigid body can be decomposed into two independent motions: a translation of the CM and a rotation about the CM.

Only motion for which the axis of rotation remains parallel to a given direction — say $z$-axis will be considered.

Want to show that the total $L_z$ can be written as the sum of two terms:

$$L_z = I_0 \omega + (\vec{R} \times m\vec{V})_z$$

$\vec{R}$: Position vector of CM
$\vec{V}$: $= \frac{d\vec{R}}{dt}$; velocity of CM

Consider body made up of $m_j$ mass elements with position vectors $\vec{r}_j$.

$$\vec{L} = \sum_j \vec{r}_j \times m_j \vec{v}_j$$
$$\vec{R} = \sum m_j \vec{r}_j$$
$$M = \text{Total Mass}$$
\[ \vec{n}_j = \vec{R} + \vec{n}'_j \]

\[ L = \sum \vec{r}_j \times \vec{m}_j \vec{v}_j \]
\[ = \sum (\vec{R} + \vec{n}'_j) \times \vec{m}_j (\vec{R} + \vec{n}'_j) \]
\[ = \vec{R} \times \sum m_j \vec{R} + \sum m_j \vec{n}'_j \times \vec{R} + \vec{R} \times \sum m_j \vec{n}'_j + \sum m_j \vec{R}_j \times \vec{v}_j \]

**Term 2:** \[ \sum m_j \vec{n}'_j = \sum m_j (\vec{n}'_j - \vec{R}) \]
\[ = \sum m_j \vec{R} - m \vec{R} \]
\[ \equiv 0 \quad \text{Def. of CM} \]

**Term 3:** \[ \sum m_j \vec{R}_j \equiv 0 \quad \text{Def. of CM Velocity in CM} \]

**Term 1:** \[ \vec{R} \times \sum m_j \vec{R} = \vec{R} \times m \vec{R} \]
\[ = \vec{R} \times m \vec{v}_{CM} \]
\[ \therefore L = \vec{R} \times m \vec{v}_{CM} + \sum m_j \vec{R}_j \times \vec{v}_j \]
First Term: \( \vec{L}_0 \): Angular momentum due to cm motion. Called orbital angular momentum.

Take a fixed axis for rotation: \( z \)-axis

\[ \sum (\vec{r}_j \times m_j \vec{v}_j) \]: \( L_z \), Spin Angular Momentum.

Body rotates about \( z \)-axis

For \( m_j \); \( |\vec{p}_j| \) a constant!

\[ \vec{v}_j \] must be \( \perp \) to \( \vec{p}_j \)

\( \perp \) distance of \( m_j \) to \( z \)-axis is \( g_j \).

\[ L_z (j) = m_j \vec{v}_j \times \text{distance to axis} \]

\[ = m_j \vec{v}_j \times g_j \]

But \( \vec{v}_j = \omega g_j \), \( \omega \): angular velocity.

\[ L_z (j) = m_j \omega g_j^2 \]

\( L_z = \sum L_z (j) = \sum m_j g_j^2 \omega \)

\[ L_z = I_z \omega \] Spin Angular Momentum.
Torques: Rotation and Translation

\[ \vec{\tau} = \sum \vec{r}_j \times \vec{f}_j \]
\[ = \sum (\vec{r}_j' + \vec{R}) \times \vec{f}_j \]
\[ = \sum (\vec{r}_j' \times \vec{f}_j) + (\vec{R} \times \vec{F}) \]

where, \( \vec{F} = \sum \vec{f}_j \); total applied force.

Consider rotation about fixed z-axis.

\[ \tau_z = \tau_0 + (\vec{R} \times \vec{F})_z \]

\( \tau_0 \) = z-component of torque about cm

We had: \( L_z = I_0 \omega + (\vec{R} \times \vec{M})_z \)

\[ \frac{dL_z}{dt} = I_0 d\omega + \frac{d}{dt}(\vec{R} \times \vec{M})_z \]
\[ = I_0 \alpha + (\vec{R} \times \vec{Ma})_z \]

\[ \tau_z = \frac{dL_z}{dt} \]

\[ \therefore \tau_0 + (\vec{R} \times \vec{F})_z = I_0 \alpha + (\vec{R} \times \vec{Ma})_z \]
\[ = I_0 \alpha + (\vec{R} \times \vec{F})_z \]
\[ \vec{F} = M\alpha \]

\[ \therefore \tau_0 = I_0 \alpha \]
Rotational motion about CM depends only on torque about CM, independent of the translational motion. Result is correct even if the axis of rotation is accelerating.