Dynamics of Uniform Circular Motion

For a particle moving with speed \( v \) in a circle of radius \( R \), the centripetal acceleration is

\[
\alpha_c = \frac{v^2}{R}
\]

If the particle has a mass \( m \), the required force is given by

\[
F_c = ma_c = \frac{mv^2}{R}
\]

Whenever a particle moves in a circle with a speed \( v \), this force must be provided by some external agent.
Motion in a Vertical Circle.

- Ball whirled in a vertical circle about point O. Motion circular but is not uniform.
- Speed increases on the way down, decreases on the way up.
- \( v \) changes continuously around the path.
- \( \cdot \cdot \cdot \) we must have \( a \parallel \) and \( a \perp \).
- Forces on ball are gravity and tension.

\[
F_{\parallel} = mg \sin \theta \quad (1)
\]

\[
F_{\perp} = T - mg \cos \theta \quad (2)
\]

The tangential acceleration:

\[
a_{\parallel} = \frac{F_{\parallel}}{m} = g \sin \theta \quad (3)
\]

\[
a_{\perp} = \frac{F_{\perp}}{m} = \frac{T - mg \cos \theta}{m} = \frac{v^2}{R} \quad (4)
\]

Solve \( 4 \) for \( T = m \left( \frac{v^2}{R} + g \cos \theta \right) \)
Lowest Point: \( \Theta = 0 \)
\( F_\parallel = 0, \ a_\parallel = 0, \) acceleration is purely radial.

\[ T = m \left( \frac{v^2}{R} + g \right) \]

Highest Point: \( \Theta = 180^\circ \)
- acceleration purely radial

\[ T = m \left( \frac{v^2}{R} - g \right) \]

If speed equals a critical value \( \bar{v}_c \), tension vanishes, \( T \equiv 0 \).

\[ 0 = m \left( \frac{\bar{v}_c^2}{R} - g \right) \]
\[ \bar{v}_c = \sqrt{Rg} \]

The speed at any point can be determined from energy considerations given a value at some initial point.
Example: Loop-the-loop

\( m = 75 \text{ kg} \)
\( R = 10 \text{ m} \)
\( v_0 = 15 \text{ m/s} \)

A man in a car executes motion in a vertical loop at a constant speed \( v_0 \).

Calculate the apparent force the man feels pressing against him at the bottom and top of the loop. i.e. Calculate the apparent weight.

**Bottom**

\[ N - mg = \frac{m v_0^2}{R} \quad \text{(Circle)} \]

Normal force pushing on man (weight)

\[
N = m \left( g + \frac{v_0^2}{R} \right) = mg \left[ 1 + \frac{1}{g} \frac{v_0^2}{R} \right] = mg \left[ 1 + \frac{1}{9.81} \frac{15^2}{10} \right] = mg \left[ 1 + 2.29 \right] = 3.29 (mg)
\]
\[ N + mg = \frac{mv_0^2}{R} \]

\[ N = mg \left[ \frac{v_0^2}{R g} - 1 \right] \]

\[ = mg \left[ 2.29 - 1 \right] \]

\[ = 1.29 (mg) \]

What happens if \( v_0 \) is decreased?
In particular what if
\[ \frac{v_0^2}{R g} = 1 \quad \Rightarrow \quad v_0 = \sqrt{g R} = \sqrt{9.81 \times 10} \]

\[ = 9.90 \text{ m/s} \]

\( N \leq 0 \); No normal force.

What if \( v_0 < \sqrt{g R} \) ??
Conical Pendulum

Object at end of rope of length $L$. Total mass is $m$. Object swings in a horizontal circle of radius $R$ at constant speed $v$.

Assume period of revolution is $\tau$.

Find tension in rope $T$ and angle $\theta$.

\[ \Sigma \vec{F} = ma \]

\[ T \sin \theta = m \frac{v^2}{R} \quad (1) \]

\[ T \cos \theta - mg = 0 \quad (2) \]

\[ \tan \theta = \frac{v^2}{Rg} \quad (3) \]

\[ R = L \sin \theta \]

\[ v = \frac{2\pi L \sin \theta}{\tau} \]

Then Eq. (3) becomes

\[ \cos \theta = \frac{g \frac{v^2}{R}}{4\pi^2 L} \]

or

\[ \tau = 2\pi \sqrt{\frac{L}{g}} \cos \theta \]
Work and Energy

- Newton's laws of motion relating acceleration to forces acting enable us to predict future values of the position and velocity of a particle.
- In this lecture we will see how to relate force to particle motion in a second way. The scalar product of force and displacement defines work. The product of mass and the square of a particle's velocity gives twice the kinetic energy.
- Combining work and K.E. we derive the work-energy principle. This principle plays a role which is analagous to that of Newton's Second Law.

\[
\text{Work-Energy Principle} \rightarrow \text{Conservation of Energy}
\]
\[
\Delta t \cdot \text{(Force)} \rightarrow \text{Linear Impulse} \rightarrow \text{Conservation of Momentum}
\]
\[
\text{Force} \otimes \text{Position} \rightarrow \text{Torque} \rightarrow \text{Conservation of Angular Momentum}
\]

- Conservation laws are closely connected to a symmetry associated with our mathematical description of the physics:
  - Momentum \(\longleftrightarrow\) Translation in Space
  - Energy \(\longleftrightarrow\) Translation in Time
  - Angular Momentum \(\longleftrightarrow\) Invariance to Rotations
  - Electric Charge \(\longleftrightarrow\) GM Shift in Phase
Work - 1 Dimension

Force $F_x$ acting on a particle moving along $x$ does an amount of work:

$$W = F_x \Delta x$$

- $\Delta x$ displacement of particle
- Work done by the force $F_x$
- Work is a scalar quantity.
- $[W] = N \cdot m = J oules$

$W > 0$ - Force and displacement are in the same direction
$W < 0$ - Force and displacement are opposed
$W = 0$ - Represents work done by the particle.

- When several forces act, add individual contributions.
- Work is dependent on the frame of reference.

Elevator Frame
- $W = 0$  $\Delta x = 0$
- $v = \text{const}$

Laboratory Frame
- $W \neq 0$  $\Delta x \neq 0$
- $v = \text{const}$
Variable Force/Work.

\[ F_x = F_x(x) \]

- Force is a function of position, spring, gravity, etc.

What is \( W(a \to b) \) in moving a particle from \( x = a \) to \( x = b \).

Divide the total displacement into a large number of small intervals \( \Delta x \).

For each interval

\[ \Delta W_i = F_x(x_i) \Delta x \Rightarrow \text{Area of a rectangle of height } F_x \text{ and width } \Delta x. \]

Total work from \( x = a \) to \( x = b \) is the sum of all such intervals.

\[
\begin{align*}
W &= \sum_{i=0}^{n-1} \Delta W_i = \sum_{i=0}^{n-1} F_x(x_i) \Delta x \\
\end{align*}
\]

In the limiting case \( \Delta x \to 0 \)

\[ n \to \infty \]

\[ W = \lim_{\Delta x \to 0} \sum_{i} F_x(x_i) \Delta x \]

\[ W = \int_{a}^{b} F_x(x) \, dx \quad \text{[Definite Integral]} \]

The total displacement from \( a \) to \( b \) has been divided into small intervals of \( \Delta x \).
Work \equiv \text{Area bounded by the curve } F_x(x) \text{ and the lines } x=a \text{ and } x=b \text{ and the } x\text{-axis.}

\text{Example: Spring Force}

F(x) = -kx

How much work is needed to move a spring (fixed at one end) from } x=a \text{ to } x=b? \text{ }
\begin{align*}
W &= \int_{a}^{b} F(x) \, dx \\
&= \int_{a}^{b} (-kx) \, dx \\
&= -\frac{kx^2}{2} \bigg|_{a}^{b} \\
&= -\frac{k(b^2-a^2)}{2}
\end{align*}

W < 0, \text{ work is done on the spring}!!

The straight line is a plot of the function $F_x(x) = -kx$. The quadrilateral area to the right of $a$, and the left of $b$ represents the work.
Work - 3 Dimensions

In general:
\[ W = F \cdot \Delta r \]
\[ \text{displacement.} \]
\[ \text{constant force} \]

\[ = F (\Delta r) \cos \theta \]
\[ W = 0 \text{ if } \vec{F} \perp \Delta r \]
\[ W = F_x (\Delta x) + F_y (\Delta y) + F_z (\Delta z) \]
\[ \text{[Using components]} \]

If \( F = F(x) \) is not constant,
\[ \Delta W = F(\Delta r) \cos \theta \]

Limiting case \( \Delta r \to 0 \)
\[ W = \int_{P_1}^{P_2} \vec{F} \cdot d\vec{r} \]
\[ \text{[Line-Integral]} \]
\[ = \int_{P_1}^{P_2} (F_x \, dx + F_y \, dy + F_z \, dz) \]
\[ = \int_{P_1}^{P_2} F_x \, dx + \int_{P_1}^{P_2} F_y \, dy + \int_{P_1}^{P_2} F_z \, dz \]

The infinitesimal displacement \( dr \) is tangent to the path.
Example: Gravitational Force

\[ F_x = 0 \quad \text{Force of Gravity} \]
\[ F_y = 0 \quad \text{close to earth's} \]
\[ F_z = -mg \quad \text{surface.} \]

\[ W = \int_{z_1}^{z_2} F_z \, dz = -mg(z_2 - z_1) \]

\[ W = -mg (\Delta z) \quad \text{change in height.} \]

Work done by gravity depends on the vertical separation between \( P_1 \) and \( P_2 \).

NB: Complication or details of the motion do not count — only \( (\Delta z) \)!!
Example

A block is pushed along a horizontal plane at constant velocity. Coefficient of sliding friction is $\mu_k$. How much work is done in a distance $s$ by the force $F$?

\[ F \cos \theta - f = 0 \quad \text{(constant } v) \]
\[ -F \sin \theta + N - mg = 0 \]

\[ f = \mu_k N = \mu_k (F \sin \theta + mg) \]

\[ F \cos \theta - \mu_k (F \sin \theta + mg) = 0 \]

\[ \therefore F = \frac{\mu_k mg}{\cos \theta - \mu_k \sin \theta} \]

\[ W_F = \int F \cdot ds \quad \text{[Constant Force]} \]

\[ = Fscos \theta = \frac{\mu_k mg s}{1 - \mu_k \tan \theta} \]

Q: What happens when $\mu_k \tan \theta = 1$ ?

\[ i.e. \quad \tan \theta = \frac{1}{\mu_k} \quad W \to \infty \quad ??? \]
\[ f = F \cos \theta \]

\[ = \frac{\mu_k mg \cos \theta}{\cos \theta - \mu_k \sin \theta} \]

\[ W_{f} = \vec{F} \cdot \vec{dS} \]

\[ = -\frac{\mu_k mg s \cos \theta}{\cos \theta - \mu_k \sin \theta} \text{ [F and } \vec{dS} \text{ are opposite]} \]

\[ W_N = 0 \quad \text{since } \vec{N} \text{ and } \vec{dS} \text{ are } \perp \]

\[ W_{mg} = 0 \quad \text{since } \vec{mg} \text{ and } \vec{dS} \text{ are } \perp \]

Total work on object:

\[ W = W_F + W_{f} + W_N = 0. \]

True since object moves at constant velocity, net force on object \( \vec{F}_R = 0 \).
Example: Work Done on an Astronaut

What is the work done by the force of gravity on an 80-kg astronaut in a displacement from point A at the earth's surface to point B whose altitude is 2R_E (earth radii)?

\[ F = -\frac{G M_E m}{r^2} \]

\[ W = -\int_0^{3R_E} \frac{G M_E m}{R_E} \frac{dr}{r^2} \]

\[ \int \frac{1}{r^2} \, dr = -\frac{1}{r} \]

\[ W = + \left. \frac{G M_E m}{R} \right|_{R_E}^{3R_E} = G M_E m \left( \frac{1}{3R_E} - \frac{1}{R_E} \right) \]

\[ = -\frac{2}{3} \frac{G M_E m}{R_E} = -\frac{2}{3} m g R_E \]

Negative work: Force directed towards earth, displacement is away from earth.

\[ W = -3.34 \times 10^9 \text{ J} \]

The force acting on an object that is moving away from the earth yields the negative shaded area.