Vectors

Vectors are mathematical quantities that are useful to describe physical quantities that have both a magnitude and direction associated with them.

We will review the operations of vector addition, subtraction and multiplication by a number. We will show two types of vector multiplication.

We will deal with vectors in two different ways:

- Geometric approach
- Algebraic approach

Want to describe motion of an ideal particle
- Choose a coordinate system - e.g. Cartesian $x,y,z$
- Want to describe the motion in a general way.
- Note that the "laws of nature" must be invariant to the choice of an inertial (non-accelerated) coordinate system.
- Consider two successive displacements.

- Start at A and move to B.
- The directed line segment from A to B is called a displacement vector. Arrow indicates direction of motion.
- Line does not represent actual path followed just the final result.
- Move from B \rightarrow C.
- \overrightarrow{AC} represents result of both moves. Net direction; Net displacement.
- "Addition of Vectors"
Vector: Any quantity that has a magnitude and direction and behaves like the displacement vector.

- Displacement
- Velocity
- Acceleration

Example:

Scalar: Any quantity that has a magnitude but no direction.

- Length
- Time
- Mass
- Area
- Volume

\[ \vec{A} = \vec{B} \]

- Only if \( |A| = |B| \) magnitude
- Only if directions are equal
- Location and starting point do not matter
- Units must be the same
**Vector Addition**

\[ \vec{A} + \vec{B} \]

\[ \vec{C} = \vec{A} + \vec{B} \]

\[ \vec{C} = \vec{B} + \vec{A} \]

\[ \therefore \vec{C} = \vec{A} + \vec{B} = \vec{B} + \vec{A} \]

↑ *Commutative Law*

- Order of addition does not matter!
- Addition makes sense only for same kinds of vectors.

**Parallelogram:**

3-or More Vectors:

\[ \vec{R} = (\vec{A} + \vec{B}) + \vec{C} \]

\[ = \vec{A} + (\vec{B} + \vec{C}) \]
Negative of Vector
- same magnitude
- opposite direction
\[ \vec{A} + (-\vec{A}) = 0 \]

Subtraction
- The subtraction of two vectors $\vec{A}$ and $\vec{B}$ is defined as the sum of $\vec{A}$ and $-\vec{B}$.
\[ \vec{C} = \vec{A} - \vec{B} = \vec{A} + (-\vec{B}) \]

Scalar x Vector
- magnitude multiplied by scalar.
- same direction
- different physical quantities: m has own units.

\[ \vec{F} = m \vec{\alpha} \]
Vector Components

- A vector is completely described by its components.
- Useful for vector algebra.
- Choose a coordinate system.
- Choose an origin at foot-of-vector.

The position vector $\mathbf{r}$ of the point $P$.

The unit vectors.

- $P(x, y, z)$ is an arbitrary point with coordinates $x, y, z$.
- $\mathbf{r}$ is a position vector from the origin to the point $x, y, z$.

Unit Vectors

$\hat{x}$, $\hat{y}$, and $\hat{z}$ are three vectors of magnitude 1 unit (dimensionless) pointing along the coordinate axes.

- Other vectors can be written in terms of them.
- Called unit vectors.
- Carry no units.
Components - 2 dimensions

\[ \vec{r} = \vec{r}_x + \vec{r}_y \]

\[ r_x = r \cos \theta \]

\[ r_y = r \sin \theta \]

- Components can be used instead of vector itself.
\[ \vec{r} = x \hat{x} + y \hat{y} \]

\[ r = \sqrt{x^2 + y^2} \]

\[ x = r \cos \theta \]

\[ y = r \sin \theta \]

\[ \tan \theta = \frac{y}{x} \]
What is

\[ \hat{x} + \hat{y} + \hat{z} = ? \]

- Move a distance \( x \)-units along \( x \)-axis
- \( y \)-units along \( y \)-axis
- \( z \)-units along \( z \)-axis

\[ \mathbf{r} = \hat{x} + \hat{y} + \hat{z} \]

**Arbitrary Vector \( \mathbf{A} \)**

- Coordinate system at the foot of vector \( \mathbf{A} \).
- Drop perpendiculars from the tip to each of the coordinate axes.
- Intercepts give the components with values (numerical)
  \( A_x, A_y, A_z \).

In terms of unit vectors

\[ \mathbf{A} = A_x \hat{x} + A_y \hat{y} + A_z \hat{z} \]

The components \( x, y, z \) of the position vector \( \mathbf{r} \).

The components \( A_x, A_y, A_z \) of an arbitrary vector \( \mathbf{A} \).
Vector A in two dimensions and its components.

Example: 2-Dimensions

\[
A_x = A \cos \theta_x
\]

\[
A_y = A \sin \theta_x
\]

magnitude of \( \vec{A} \):

\[
|\vec{A}| = A = \sqrt{A_x^2 + A_y^2}
\]

\[
\vec{A} = A_x \hat{x} + A_y \hat{y}
\]

\[
= A \cos \theta_x \hat{x} + A \sin \theta_x \hat{y}
\]

Example

Want: \( \vec{A} + \vec{B} = ? \)

\[
|A| = 3 \quad \{ \text{Given magnitudes} \}
\]

\[
|B| = 4
\]

- Select most convenient coordinate system
- Choose one axis along one of the vectors.
Want:
Magnitude of \( \vec{R} \)
Direction of \( \vec{R} \)

\[
\vec{A} = A_x \hat{\text{i}} + A_y \hat{\text{j}} = A \hat{\text{i}} + 0 \hat{\text{j}}
\]

\[
\vec{B} = B_x \hat{\text{i}} + B_y \hat{\text{j}} = B \cos 60^\circ \hat{\text{i}} + B \sin 60^\circ \hat{\text{j}}
\]

\[
\vec{R} = \vec{A} + \vec{B} = (A_x + B_x) \hat{\text{i}} + (A_y + B_y) \hat{\text{j}} = (A + B \cos 60^\circ) \hat{\text{i}} + (B \sin 60^\circ) \hat{\text{j}}
\]

\[
\vec{R} = 5 \hat{\text{i}} + 3.46 \hat{\text{j}}
\]

\[
|\vec{R}| = \sqrt{R_x^2 + R_y^2} = \sqrt{5^2 + 3.46^2} = 6.08
\]

\[
\alpha = \tan^{-1} \frac{R_y}{R_x} = \tan^{-1} \frac{3.46}{5} = 34.7^\circ
\]
Example - General 3-D Vector

\[ A_x = A \cos \alpha \]
\[ A_y = A \cos \beta \]
\[ A_z = A \cos \gamma \]

\[ \alpha, \beta, \gamma \Rightarrow \text{direction cosines} \]

\[ \cos^2 \alpha + \cos^2 \beta + \cos^2 \gamma = 1 \]
(not all independent)

Magnitude: \[ A = \sqrt{A_x^2 + A_y^2 + A_z^2} \]

\[ \cos \alpha = \frac{A_x}{A}, \text{ etc.} \]

Position vector \( \vec{r} \)

\[ \vec{r} = x \hat{i} + y \hat{j} + z \hat{k} \]

\[ r = \sqrt{x^2 + y^2 + z^2} \]

Distance from \((x, y, z)\) to origin.

\[ \vec{A} = A \cos \alpha \hat{i} + A \cos \beta \hat{j} + A \cos \gamma \hat{k} \]
Let
\[ \vec{A} = A_x \hat{x} + A_y \hat{y} + A_z \hat{z} \]
\[ \vec{B} = B_x \hat{x} + B_y \hat{y} + B_z \hat{z} \]

\[ \vec{A} + \vec{B} = (A_x + B_x) \hat{x} + (A_y + B_y) \hat{y} + (A_z + B_z) \hat{z} \]
\[ \vec{A} - \vec{B} = (A_x - B_x) \hat{x} + (A_y - B_y) \hat{y} + (A_z - B_z) \hat{z} \]

Generalize to any number of vectors added or subtracted from each other.

Alternate Notation

\[ \vec{A} = A_x \hat{i} + A_y \hat{j} + A_z \hat{k} \]
**Inclined Plane**

\[ \vec{B} \]

\[ \theta = 30^\circ \]

Decompose vector \( \vec{B} \):

\( B_{\parallel} \): Parallel to \( x \).

\( B_{\perp} \): Perpendicular to \( x \); i.e. along \(-y\).

\[ \vec{B} = B_{\parallel} \hat{\vec{x}} + B_{\perp} \hat{\vec{y}} \]

\[ B_{\parallel} = |\vec{B}| \cos 30^\circ \]

\[ B_{\perp} = -|\vec{B}| \sin 30^\circ \]
Vector Multiplication

- Several ways to multiply vectors
- Need to take into account magnitude and direction

1. Dot Product
   - scalar product
   - inner product

Vectors: \( \vec{A} \) and \( \vec{B} \)

\[ \vec{A} \cdot \vec{B} = AB \cos \phi \quad \phi \leq 180^\circ \]

- Result is a scalar number
  \[ \vec{A} \cdot \vec{B} = \vec{B} \cdot \vec{A} \quad \text{(commutative)} \]

\[ \vec{A} \cdot \vec{A} = AA \cos 0^\circ \]

= \( A^2 \)

[parallel vectors]

= (magnitude)\(^2\)

\[ \vec{A} \cdot (-\vec{A}) = -A^2 \]

[antiparallel vectors]
\[ \vec{a} \cdot \vec{b} = A (B \cos \phi) \]
\[ \text{Proj. of } \vec{b} \text{ on } \vec{a} \]
\[ = B (A \cos \phi) \]
\[ \text{Proj. of } \vec{a} \text{ on } \vec{b} \]

The component of A along B is \( A \cos \phi \)
the component of B along A is \( B \cos \phi \).

\[ 0 < \phi < 90^\circ \quad (\vec{a} \cdot \vec{b}) > 0 \]
\[ 90 < \phi < 180^\circ \quad (\vec{a} \cdot \vec{b}) < 0 \]

\[ \phi = 90^\circ \quad (\vec{a} \cdot \vec{b}) = 0 \]
- \( \vec{a} \) and \( \vec{b} \) are \( \perp \) to each other
- A good test for \( \perp \) vectors.

\[ \hat{\vec{a}} \cdot \hat{\vec{a}} = 1 \]
\[ \hat{\vec{a}} \cdot \hat{\vec{b}} = 1 \]
\[ \hat{\vec{a}} \cdot \hat{\vec{b}} = 1 \]
\[ \text{Parallel unit vectors} \]

\[ \hat{\vec{a}} \cdot \hat{\vec{a}} = \hat{\vec{b}} \cdot \hat{\vec{b}} = 0 \]
\[ \text{Perpendicular unit vectors} \]
Distributive Law

To work out product of \( \vec{A} \cdot \vec{B} \) we make use of the fact that vector multiplication obeys the 'distributive law':

\[
(\vec{C} + \vec{D}) \cdot \vec{E} = \vec{C} \cdot \vec{E} + \vec{D} \cdot \vec{E}
\]

Proof: Product is the magnitude of \( \vec{E} \) times component \( \vec{C+D} \) along \( \vec{E} \). But the component of \( \vec{C+D} \) along any direction is equal to the sum of the component of \( \vec{C} \) plus the component of \( \vec{D} \).
\[ \vec{A} \cdot \vec{B} = (A_x \hat{x} + A_y \hat{y} + A_z \hat{z}) \cdot (B_x \hat{x} + B_y \hat{y} + B_z \hat{z}) \]
\[ = A_x B_x \hat{x} \cdot \hat{x} + A_x B_y \hat{x} \cdot \hat{y} + A_x B_z \hat{x} \cdot \hat{z} \]
\[ + A_y B_x \hat{y} \cdot \hat{x} + A_y B_y \hat{y} \cdot \hat{y} + A_y B_z \hat{y} \cdot \hat{z} \]
\[ + A_z B_x \hat{z} \cdot \hat{x} + A_z B_y \hat{z} \cdot \hat{y} + A_z B_z \hat{z} \cdot \hat{z} \]

\[ \vec{A} \cdot \vec{B} = A_x B_x + A_y B_y + A_z B_z \]

Each term is a product of vectors which are either \(\parallel\) or \(\perp\).

What is
\[ \hat{y} \cdot \vec{A} = \hat{y} \cdot [A_x \hat{x} + A_y \hat{y} + A_z \hat{z}] \]
\[ = A_y \parallel \]

Component of \(A\) along \(y\)-axis.

First application of a scalar product will be the concept of work. The work done for a constant force \(\vec{F}\) acting on a body which is displaced an amount \(\vec{d}\), is given by
\[ W = \vec{F} \cdot \vec{d} \]
Example: [Dot Product]

\[ \vec{A} = 3\hat{x} + 7\hat{y} \]

\[ \vec{B} = -\hat{x} + 2\hat{y} + \hat{z} \]

\[ A_x = 3 \quad B_x = -1 \]
\[ A_y = 0 \quad B_y = 2 \]
\[ A_z = 7 \quad B_z = 1 \]

\[ \vec{A} \cdot \vec{B} = A_xB_x + A_yB_y + A_zB_z \]
\[ = (3)(-1) + (0)(2) + (7)(1) \]
\[ = +4 \]

\[ \vec{A} \cdot \vec{B} = 1A|B| \cos \theta \]

\[ \cos \theta = \frac{\vec{A} \cdot \vec{B}}{AB} = \frac{4}{\sqrt{3^2 + 0^2 + 7^2} \sqrt{(-1)^2 + 2^2 + 1^2}} = \frac{4}{\sqrt{58} \sqrt{6}} \]

\[ \theta = 77.6° \quad \text{[Angle between two vectors } \vec{A}, \vec{B} \text{]} \]

\[ \uparrow \text{ Easy Way} !! \]