Elastic Collisions - Two Dimensions

- Assume special case where target particle is initially at rest, $\vec{v}_a = 0$.
- Particle collide by interacting. For many forces, gravity, electromag., etc. forces act along line joining the particles.
- The initial particle and scattered particles define a common plane - 2-dimensional problem.

Given initial velocities, we have four unknowns following collision:

\[
\begin{align*}
\vec{v}_1' &\quad, \quad \vec{v}_2' \\
\vec{v}_{1x}' &\quad, \quad \vec{v}_{1y}' \\
\vec{v}_{2x}' &\quad, \quad \vec{v}_{2y}'
\end{align*}
\]

Particles -1 and -2.

(a) Before the collision

(b) After the collision

Schematic representation of an elastic glancing collision between two particles: (a) before the collision and (b) after the collision. Note that the impact parameter $b$, must be greater than zero for a glancing collision.
Cons. of Momentum: 2 equations (x, y)

Cons. of Energy: 1 equation

Can only get restrictions on the final motion or at least one other quantity must be known.

Cons of Momentum:
\[ m_1 \vec{v}_1 = m_1 \vec{v}_1' + m_2 \vec{v}_2' \]
\[ m_1 v_1 = m_1 v_1' \cos \theta + m_2 v_2' \cos \phi \]
\[ 0 = m_1 v_1' \sin \theta - m_2 v_2' \sin \phi \]

Cons. of Energy:
\[ \frac{1}{2} m_1 v_1^2 = \frac{1}{2} m_1 v_1'^2 + \frac{1}{2} m_2 v_2'^2 \]

Special Case: \( m_1 = m_2 \)

\[ v_1^2 = v_1'^2 + v_2'^2 \] \( \text{0} \)

\[ v_2' \cos \phi = v_1 - v_1' \cos \theta \] \( \text{2} \)

\[ v_2' \sin \phi = v_1' \sin \theta \] \( \text{3} \)
\[ (6) + (3) \quad v_a^2 = v_1^2 - 2v_1v_1' \cos \theta + v_1'^2 \]

Use Eq. (6) to eliminate \( v_a' \)

\[ v_1' = v_1 \cos \theta \]

\[ \frac{1}{2} m v_1'^2 = \frac{1}{2} m v_1^2 \cos^2 \theta \quad \text{KE of deflected incident projectile after scattering.} \]

We had cons. of momentum:

\[ \vec{v}_1 = \vec{v}_1' + \vec{v}_2' \]

\[ v_1^2 = v_1'^2 + v_2'^2 + 2v_1'v_2' \cos (\theta + \phi) \quad \text{[square]} \]

Comparing with Eq. (6), we must have that

\[ \theta + \phi = \pi/2 \]

When particles of equal mass collide, the sum of their scattering angles is 90°.

If \( m_1 > m_2 \quad \theta \leq \pi/2 \)

\[ m_1 < m_2 \quad 0 \leq \theta \leq \pi \]
**Example: 2-D Elastic Collision**

\[ m_A = 5\text{ kg} \quad \dot{v}_A = 4\text{ m/s} \]

\[ m_B = 3\text{ kg} \quad \dot{v}_{B1} = 0 \quad \text{[Stationary]} \]

Assume after collision, \( \dot{v}_{A2} = 2\text{ m/s} \)

Find \( \dot{v}_{B2} \), \( \theta \) and \( \phi \)

Since collision is elastic

\[ KE_f = KE_i \]

\[
\frac{1}{2} (5\text{ kg})(4\text{ m/s})^2 = \frac{1}{2} (5\text{ kg})(2\text{ m/s})^2 + \frac{1}{2} (3\text{ kg})\dot{v}_{B2}^2
\]

Solving,

\[ \dot{v}_{B2} = 4.47\text{ m/s} \]

Conservation of \( x \)- and \( y \)-components of momentum gives,

1. \( (5\text{ kg})(4\text{ m/s}) = (5\text{ kg})(2\text{ m/s}) \cos \theta + (3\text{ kg})(4.47\text{ m/s}) \cos \phi \)

2. \( 0 = (5\text{ kg})(2\text{ m/s}) \sin \theta - (3\text{ kg})(4.47\text{ m/s}) \sin \phi \)

Solve Eq.1 for \( \cos \phi \) \}

Square and add \( \sin^2 \phi + \cos^2 \phi = 1 \)

Then solve for \( \theta \) and finally \( \phi \):

\[ \theta = 36.7^\circ \]

\[ \phi = 26.6^\circ \]
A method used to measure the speed of a projectile such as a bullet.

**Bullet:**  \[ \text{mass} = m \]
\[ \text{speed} = v_i \] (initially)

**Block:**  \[ \text{mass} = M \gg m \]
\[ \text{speed} = 0 \] (initially)

After collision, mass \((m+M)\) moves up a height \(h\).

Collision in two parts:
1) Collision and bullet stopped in block.
2) Motion of block and bullet to maximum height \(h\).
   - Recoil

2) Collision:
   - collision time is short
   - block does not move during collision
   - no net external force during collision which is perfectly inelastic and momentum is conserved.
   - The velocity right after the collision is given by:
\[ mv_i = (m+M) v_f \]  \hspace{1cm} (1)

2. Recoil:
- After collision, the system \((m+M)\) has a kinetic energy.
- Energy is conserved.
- KE at the bottom is transformed to PE in the block and the bullet at the height \(h\).

\[
\frac{1}{2} (m+M) v_f^2 = (m+M) g h
\]

\[
\therefore v_f = \sqrt{2gh} \hspace{1cm} (2)
\]

From (1)
\[
v_i = \frac{(m+M) \sqrt{2gh}}{m}
\]

For \(\theta \ll 1\)
\[
\Delta x = L \sin \theta \approx L \theta \implies \theta \approx \frac{\Delta x}{L}
\]

\[
h = L (1 - \cos \theta) \approx \frac{L \theta^2}{2} \approx \frac{(\Delta x)^2}{2L}
\]

\[
v_i = \left( \frac{m+M}{m} \right) \sqrt{\frac{2g}{L}} \left( \frac{\Delta x}{L} \right)
\]

\[
= \left( \frac{m+M}{m} \right) \sqrt{\frac{g}{L}} (\Delta x)
\]

\[
m = 2.7 \text{ gm} \quad L = 1.14 \text{ m}
\]

\[
M = 3840 \text{ gm} \quad \Delta x = 6.5 \text{ cm}
\]

\[
v_i = 293 \text{ m/s}
\]
Ballistic Pendulum / Kinetic Energies

\[ K_i = \frac{1}{2} m v_i^2 \]
\[ K_f = \frac{1}{2} (m+M) v_f^2 = \frac{1}{2} (m+M) \left( \frac{m}{m+M} \right)^2 v_i^2 = \frac{m^2}{2(m+M)} v_i^2 \]
\[ \frac{K_f}{K_i} = \frac{m}{m+M} \ll 1 \]

Most of the initial KE is lost.

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Collision Time

Assume bullet decelerates uniformly in a distance of 0.10 m.

\[ v_f^2 - v_0^2 = 2aS \quad v_f \approx 0.0 \]

\[ a = \frac{v_0^2}{2S} = \frac{-300^2}{2 \times 0.10} \quad \text{m/s}^2 \]

Also \( v_f = v_0 + at \)

\[ t \approx \frac{-v_0}{a} = \frac{-300}{\frac{-300^2}{2 \times 0.10}} = 0.00061 \text{ s} \]

Period of pendulum:

\[ T = 2\pi \sqrt{\frac{L}{g}} = 2\pi \sqrt{\frac{1.14}{9.81}} = 2.14 \text{ s} \]

\[ t \ll T \text{ [i.e., good approx.] } \]
COLLISIONS

1. Conditions:
   An event is a collision if $\Delta t \ll \Delta T$
   - time can be separated into before, during, after
   - at collision time
   - at observation time
   An event is a collision if $|I_{ext}| < |I_{coll}|$ - the impulse of external forces can be neglected and momentum is conserved.

2. Collision Classifications:
   Elastic - KE is conserved
   Inelastic - KE is not conserved
   Completely inelastic - particles stick together after

3. Notation
   $m_1, m_2$ - masses of the two particles
   $\vec{v}_{i1}, \vec{v}_{i2}$ - initial (before collision) velocities of parts 1, 2.
   $\vec{v}_{f1}, \vec{v}_{f2}$ - final (after collision) velocities of particles 1, 2.

4. Equations
   Conservation of momentum - valid for all collisions:
   \[ m_1 \vec{v}_{i1} + m_2 \vec{v}_{i2} = m_1 \vec{v}_{f1} + m_2 \vec{v}_{f2} \]

   Conservation of kinetic energy - valid only for elastic collisions:
   \[ \frac{1}{2} m_1 v_{i1}^2 + \frac{1}{2} m_2 v_{i2}^2 = \frac{1}{2} m_1 v_{f1}^2 + \frac{1}{2} m_2 v_{f2}^2 \]
Center-of-Mass Frame/ Collisions/ KE

- 2 Particle System

\[ \vec{u}_1, \vec{u}_2 : \text{CM velocities} \]
\[ \vec{v}_1, \vec{v}_2 : \text{lab velocities} \]
\[ \vec{v}_{cm} : \text{Velocity of CM.} \]

We had for the Kinetic Energy

\[ K = \frac{1}{2} \frac{M v_{cm}^2}{m_1} + K_{INT} \]
\[ = \frac{1}{2} M v_{cm}^2 + \frac{1}{2} m_1 u_1^2 + \frac{1}{2} m_2 u_2^2 \]

Internal Energy

\[ u_1, u_2 \text{ velocities of particles relative to CM frame} \]
\[ \frac{1}{2} M v_{cm}^2 \rightarrow \text{Translational motion of CM. When no external forces act must be conserved.} \]

\[ \vec{u}_1 = \vec{v}_1 - \vec{v}_{cm} = \frac{v_1 - m_1 v_1 + m_2 v_2}{m_1 + m_2} = \frac{m_2}{m_1 + m_2} (\vec{v}_1 - \vec{v}_2) \]
\[ \vec{u}_2 = \vec{v}_2 - \vec{v}_{cm} = \frac{v_2 - m_1 v_1 + m_2 v_2}{m_1 + m_2} = -\frac{m_1}{m_1 + m_2} (\vec{v}_1 - \vec{v}_2) \]

\[ K = \frac{1}{2} M v_{cm}^2 + \frac{1}{2} \frac{m_1 m_2}{m_1 + m_2} (\vec{v}_1 - \vec{v}_2)^2 \]

↑ Relative velocity.

2nd Term: Internal Energy Represents maximum energy available for a totally inelastic collision.
\[
\text{let } \vec{v}_{rel} = \vec{u}_1 - \vec{u}_2 \]
\[
= \vec{v}_1 - \vec{v}_2 \]
\[
\text{relative velocity of two particles}
\]
\[
K = \frac{1}{2} m v_{cm}^2 + \frac{1}{2} \mu v_{rel}^2
\]
\[
\mu = \frac{m_1 m_2}{m_1 + m_2} \quad \Rightarrow \text{reduced mass for a 2-particle system.}
\]

**Collisions**

In any collision, whether elastic or inelastic, when external forces can be neglected, total momentum is conserved.

\[
\vec{P} = m_1 \vec{v}_1 + m_2 \vec{v}_2 \quad \quad \text{[constant]}
\]

Value of \( \vec{P} \) depends on coordinate system, but conservation is true in all frames.

**2-Particles**

\[
\vec{r}_{cm} = \frac{m_1 \vec{r}_1 + m_2 \vec{r}_2}{m_1 + m_2}
\]
\[
\vec{V}_c = \frac{m_1 \vec{v}_1 + m_2 \vec{v}_2}{m_1 + m_2} \quad \quad \text{(on line joining, } \vec{v}_1, \vec{v}_2)\]
Momentum in C-system:

\[ \vec{P}_{1e} = m_1 \vec{u}_1 = \frac{m_1 m_2}{m_1 + m_2} (\vec{v}_1 - \vec{v}_2) = \mu \vec{V}_{rel} \]

\[ \vec{P}_{2e} = m_2 \vec{u}_2 = -\frac{m_1 m_2}{m_1 + m_2} (\vec{v}_1 - \vec{v}_2) = -\mu \vec{V}_{rel} \]

Total Momentum:

\[ \vec{P}_C = \vec{P}_{1e} + \vec{P}_{2e} = 0. \quad \text{[cm-frame]} \]

\[ \vec{P}_C = \vec{P}_1 + \vec{P}_2 = (m_1 + m_2) \vec{V}_{cm} \]
What does collision look like in CM frame?

- Initial and final velocities determine the scattering plane.
- Each particle is scattered through the same angle $\theta$.

**Elastic:**

\[ \vec{p}_{iec} \rightarrow V_{cm} \rightarrow \vec{p}_{iec} \]

[cm is moving at constant speed before, during and after collision]

\[ |\vec{p}_{iec}| = |\vec{p}_{iec}'| \]
\[ |\vec{p}_{iac}| = |\vec{p}_{iac}'| \]

lengths of momentum vectors before and after collision are equal. Energy conserved. Collisions are back-to-back. $\vec{P}^2 = 0$.

**Inelastic**

Back-to-Back
Momentum magnitudes are reduced following collision. Loss in KE

Inelastic Collision.