CHAPTER 13  PARTIAL DERIVATIVES

13.1 Surfaces and Level Curves

The graph of $z = f(x,y)$ is a surface in three-dimensional space. The level curve $f(x,y) = 7$ lies down in the base plane. Above this level curve all points at height $7$ in the surface. The plane $z = 7$ cuts through the surface at those points. The level curves $f(x,y) = c$ are drawn in the $xy$ plane and labeled by $c$. The family of labeled curves is a contour map.

For $z = f(x,y) = z^2 - y^2$, the equation for a level curve is $x^2 - y^2 = c$. This curve is a hyperbola. For $z = z - y$ the curves are straight lines. Level curves never cross because $f(x,y)$ cannot equal two numbers $c$ and $c'$. They crowd together when the surface is steep. The curves tighten to a point when $f$ reaches a maximum or minimum. The steepest direction on a mountain is perpendicular to the level curve.

3 $x$ derivatives $\pm 1, -2, -4e^{-x}$ (flattest)
5 Straight lines
7 Logarithm curves
9 Parabolas
11: $f = (x + y)^n$ or $(ax + by)^n$ or any function of $ax + by$
13 $f(x,y) = 1 - x^2 - y^2$
15 Saddle
17 Ellipses $4x^2 + y^2 = c^2$
19 Ellipses $5x^2 + y^2 = c^2 + 4cz + x^2$
21 Straight lines not reaching $(1,2)$
23 Center $(1,1); f = z^2 + y^2 - 1$
25 Four, three, planes, spheres
27 Less than 1, equal to 1, greater than 1
29 Parallel lines, hyperbolas, parabolas
31 $\frac{dz}{dx} = 48x - 3x^2 = 0, z = 16$ hours
33 Plane; planes; 4 left and 3 right (3 pairs)

2 Level curves are circles for any function of $x^2 + y^2$; the maximum is at $(0,0)$; the functions equal 1 when $x^2 + y^2 = 3, 1, 2$, etc. (radius is square root: increasing order $f_2, f_3, f_1, f_4$).

$\frac{dz}{dx}\sqrt{3 - z^2} = -\frac{x}{2\sqrt{3 - z^2}}$ at $x = 1; \frac{dy}{dx} = \frac{y}{\sqrt{3 - z^2}}$. Similarly $\frac{dy}{dx} = -\frac{y}{2\sqrt{3 - z^2}}$ and $\frac{dz}{dx} = -\frac{x}{2\sqrt{3 - z^2}} - 1 = -2e^{-2}$.

6 $(x + y)^2 = 0$ gives the line $y = -x$; $(x + y)^2 = 1$ gives the pair of lines $x + y = 1$ and $x + y = -1$; similarly $x + y = \sqrt{2}$ and $x + y = -\sqrt{2}$; no level curve $(x + y)^2 = -4$.

8 $\sin(x - y) = 0$ on an infinite set of parallel lines $x - y = 0, \pm \pi, \pm 2\pi, \cdots$; for $c = 1$ the level curves $\sin(x - y) = 1$ are parallel lines $x - y = \frac{\pi}{2} + 2\pi n$; no level curves for $c = 2$ and $c = -4$.

10 The curve $\frac{dy}{dx} = 0$ is the axis $y = 0$ excluding $(0,0)$; $\frac{dx}{dy} = 1$ or 2 or $-4$ is a parabola.

12 $f(x,y) = zy - 1$ has level curve $f = 0$ as two pieces of a hyperbola.

14 $f(x,y) = \sin(x + y)$ is zero on infinitely many lines $x + y = 0, \pm \pi, \pm 2\pi, \cdots$

16 $f(x,y) = \max$ of $x^2 + y^2 - 1$ and zero is zero inside the unit circle.

18 $\sqrt{4x^2 + y^2} = c + 2x$ gives $4x^2 + y^2 = c^2 + 4cx + 4x^2$ or $y^2 = c^2 + 4cx$. This is a parabola opening to the left or right.

20 $\sqrt{3x^2 + y^2} = c + 2x$ gives $3x^2 + y^2 = c^2 + 4cx + 4x^2$ or $y^2 - x^2 = c^2 + 4cx$. This is a hyperbola.

26 Since $x^2 + y^2$ is always $\geq 0$, the surface $x^2 + y^2 = z^2 - 1$ has no points with $z^2$ less than 1.

30 Direct approach: $xy = \frac{(x_1y_1 + x_2y_2)}{x_1 + x_2}$ is $\frac{1}{4}(x_1y_1 + x_2y_2) + \frac{1}{2}(x_1y_1 + x_2y_2) = \frac{1}{4}(1 + 1 + \frac{1}{x_1} + \frac{1}{x_2})$

$= 1 + \frac{(x_1y_1 + x_2y_2)}{4} \geq 1$. Quicker approach: $y = \frac{x}{x}$. This is concave up (or convex) because $y'' = \frac{2}{x^2}$ is positive.

Note for convex functions: Tangent lines below curve, secant line segments above curve!

$y = \frac{2048}{x}$ has $\frac{dy}{dx} = \frac{-4096}{x^2} = -1$ at $x = 16$. Also $y'' = \frac{12288}{x^4} > 0$ so the curve is concave up (or convex).

The line $x + y = 24$ also goes through $(6,8)$ with slope $-1$; it must be the tangent line.
34 The function \( f(x, y) \) is the height above the ground. The level curve \( f = 0 \) is the outline of the shoe.

### 13.2 Partial Derivatives

The partial derivative \( \partial f / \partial y \) comes from fixing \( x \) and moving \( y \). It is the limit of \( f(x, y + \Delta y) - f(x, y) \)/\( \Delta y \). If \( f = e^{2x} \sin y \) then \( \partial f / \partial y = 2e^{2x} \cos y \). If \( f = (x^2 + y^2)^{1/2} \) then \( f_x = y/(x^2 + y^2)^{1/2} \) and \( f_y = x/(x^2 + y^2)^{1/2} \). At \((x_0, y_0)\) the partial derivative \( f_x \) is the ordinary derivative of the partial function \( f(x, y) \). Similarly \( f_y \) comes from \( f(x_0, y) \). Those functions are cut out by vertical planes \( z = z_0 \) and \( y = y_0 \), while the level curves are cut out by horizontal planes.

The four second derivatives are \( f_{xx}, f_{xy}, f_{yx}, f_{yy} \). For \( f = xy \) they are \( 0,1,1,0 \). For \( f = \cos 2x \cos 3y \) they are \(-4 \cos 2x \cos 3y, 6 \sin 2x \sin 3y, -9 \cos 2x \cos 3y \). In those examples the derivatives \( f_{xy} \) and \( f_{yx} \) are the same. That is always true when the second derivatives are continuous.
The tangent plane to \( y = f(x) \) is \( y - y_0 = f'(x_0)(x - x_0) \). The tangent plane to \( w = f(x,y) \) is \( w - w_0 = (\partial f/\partial x)_0(x - x_0) + (\partial f/\partial y)_0(y - y_0) \). The normal vector is \( \mathbf{N} = (f_x, f_y, -1) \). For \( w = x^3 + y^3 \) the tangent equation at \((1,1,2)\) is \( w - 2 = 3(x - 1) + 3(y - 1) \). The normal vector is \( \mathbf{N} = (3, 3, -1) \). For a sphere, the direction of \( \mathbf{N} \) is out from the origin.

The surface given implicitly by \( F(x,y,z) = c \) has tangent plane with equation \( (\partial F/\partial x)_0(x - x_0) + (\partial F/\partial y)_0(y - y_0) + (\partial F/\partial z)_0(z - z_0) = 0 \). For \( xyz = 6 \) at \((1,2,3)\) the tangent plane has the equation \( 6(x-1) + 3(y-2) + 2(z-3) = 0 \). On that plane the differentials satisfy \( 6dx + 3dy + 2dz = 0 \). The differential of \( z = f(x,y) \) is \( dz = f_x dx + f_y dy \). This holds exactly on the tangent plane, while \( \Delta z \approx f_x \Delta x + f_y \Delta y \) holds approximately on the surface. The height \( z = 3x + 7y \) is more sensitive to a change in \( y \) than in \( x \).
because the partial derivative $\frac{\partial z}{\partial y} = 7$ is larger than $\frac{\partial z}{\partial x} = 3$.

The linear approximation to $f(x, y)$ is $f(x_0, y_0) + (\frac{\partial f}{\partial x})(x - x_0) + (\frac{\partial f}{\partial y})(y - y_0)$. This is the same as $\Delta f \approx (\frac{\partial f}{\partial x})\Delta x + (\frac{\partial f}{\partial y})\Delta y$. The error is of order $(\Delta x)^2 + (\Delta y)^2$. For $f = \sin zy$ the linear approximation around $(0, 0)$ is $f(x, y) = 0$. We are moving along the tangent plane instead of the surface. When the equation is given as $F(x, y, z) = c$, the linear approximation is $F_x \Delta x + F_y \Delta y + F_z \Delta z = 0$.

Newton's method solves $g(x, y) = 0$ and $h(x, y) = 0$ by a linear approximation. Starting from $x_n, y_n$ the equations are replaced by $g_x \Delta x + g_y \Delta y = -g(x_n, y_n)$ and $h_x \Delta x + h_y \Delta y = -h(x_n, y_n)$. The steps $\Delta x$ and $\Delta y$ go to the next point $(x_{n+1}, y_{n+1})$. Each solution has a basin of attraction. Those basins are likely to be fractals.
12 \( N_1 = 2i + 4j - k \) and \( N_2 = 2i + 6j - k \) give \( v = \begin{vmatrix} 1 & j & k \\ 2 & 4 & -1 \\ 2 & 6 & -1 \end{vmatrix} = 2i + 4k \) tangent to both surfaces.

14 The direction of \( N \) is \( 2xy^2 + 2x^2y - k = 8i + 4j - k \). So the line through \((1,2,4)\) has \( x = 1 + 8t, y = 2 + 4t, z = 4 - t \).

16 The normal line through \((x_0, y_0, z_0)\) has direction \( N = (F_x)i + (F_y)j + (F_z)k \). This is the radial line from the origin if \((F_x) = cx_0, (F_y) = cy_0, (F_z) = cz_0 \). Then \( F \) is a function of \( x^2 + y^2 + z^2 \) and the surface is a sphere.

18 \( df = yz \, dx + xz \, dy + xy \, dz \).

20 Direct method: \( R = \frac{R_1R_2}{R_1 + R_2} \) and \( \frac{\partial R}{\partial x} = \frac{R_2^2}{(R_1 + R_2)^2} \) and \( \frac{\partial R}{\partial y} = \frac{R_1^2}{(R_1 + R_2)^2} \). If \( R_1 = 1 \) and \( R_2 = 2 \) then \( \frac{\partial R}{\partial x} \) is four times larger \((\frac{4}{9} \text{ vs.} \frac{1}{9}) \); more sensitive than \( R_1 \). By chain rule: \(-\frac{1}{R_1^2} \frac{\partial R}{\partial x} = -\frac{1}{R_1} \) and \(-\frac{1}{R_2^2} \frac{\partial R}{\partial y} = -\frac{1}{R_2} \).

22 (a) Common sense: 2 hits raises an average that is below \( \mu \). Mathematics:

\[
A = \frac{x}{y} \text{ has } dA = \frac{x \, dx - y \, dy}{y^2} > 0 \text{ if } y \, dx > x \, dy. \text{ This is again } \frac{dx}{dy} > \frac{y}{x} \text{ or } A > A. \text{ (b) The player has } x = 200 \text{ hits since } 0.005 = .5. \text{ We want to choose } \Delta x = \Delta y \text{ (all hits) so } \Delta A \text{ reaches } .005. \text{ But } \Delta A \approx \frac{\Delta x \Delta y}{\Delta x^2} = \frac{200}{400} = .005 \text{ when } \Delta x = 4 \text{ hits. Check: } 204 \approx 505 \text{ (to 3 decimals)}. \text{ If averages are rounded down we need } \Delta x = 5 \text{ hits.}

24 (1) \( c \) is between \( x_0 \) and \( x_0 + \Delta x \) by the Mean Value Theorem (2) \( C \) is between \( y_0 \) and \( y_0 + \Delta y \) (3) the limit exists if \( f_x \) is continuous (4) the limit exists if \( f_y \) is continuous.

26 \( P = \frac{4a^2 + 2t}{s^2 + 2} \) and \( Q = \frac{40 - t}{s^2 + 2} \) so \( \frac{\partial P}{\partial s} = \frac{8 - 2t}{(s^2 + 2)^2} \) and \( \frac{\partial P}{\partial t} = \frac{2}{s^2 + 2} \). At \( s = 4, t = 10 \) this gives \( \frac{\partial P}{\partial s} = \frac{6}{9} \) and \( \frac{\partial P}{\partial t} = \frac{2}{62} \).

28 Take partial derivatives with respect to \( b \): \( 2x \frac{\partial b}{\partial b} + b \frac{\partial x}{\partial b} + x = 0 \) or \( \frac{\partial x}{\partial b} = -\frac{x}{2x + b} \). Similarly \( 2x \frac{\partial b}{\partial c} + b \frac{\partial x}{\partial c} + 1 = 0 \) gives \( \frac{\partial x}{\partial c} = -\frac{1}{2x + b} \). Then \( \frac{\partial x}{\partial b} \) is larger (in magnitude) when \( x = 2 \).

30 (a) The third surface is \( z = 0 \). (b) Newton uses the tangent plane to the graph of \( g \), the tangent plane to the graph of \( h \), and \( z = 0 \).

32 \( \frac{3}{4} \Delta x - \Delta y = \frac{3}{8} \) and \( -\Delta x + \frac{3}{4} \Delta y = -\frac{3}{8} \). The new point is \((-1, -1)\), an exact solution. The point \((\frac{3}{4}, \frac{3}{4})\) is in the gray band (upper right in Figure 13.11a) or the blue band on the front cover of the book.

34 \( 3a^2 \Delta x - 2x \Delta y = -a^3 \) and \( -\Delta x + 0 \Delta y = a \) give \( \Delta x = -a \) and \( \Delta y = -2a^3 \). The new point is \((0, -2a^3)\) on the \( y \) axis. Then \( 0 \Delta x - \Delta y = -2a^3 \) and \( -\Delta x + 3(4a^6) \Delta y = 8a^9 \) give \( \Delta y = 2a^3 \) and \( \Delta x = 16a^9 \). The new point \((16a^9, 0)\) is the same as the start \((a, 0)\) if \( 16a^9 = 1 \) or \( a = \pm \frac{1}{a} \). In these cases Newton's method cycles. Question: Is this where the white basin ends along the \( x \) axis?

36 By Problem 34 Newton's method diverges if \( 16a^9 > 1 \): for instance \((x_0, y_0) = (1, 0)\) as in Example 9 in the text.

38 A famous fractal shows the three basins of attraction – see almost any book displaying fractals. Remarkable property of the boundary points between basins: they touch all three basins! Try to draw 3 regions with this property.

40 Problem 39 has \( 2x \Delta x - \Delta y = y - x^2 \) and \( \Delta x - \Delta y = y - x \). Subtraction gives \( (2x - 1) \Delta x = x - x^2 \). Then \( x + \Delta x = x + \frac{2 - x}{2x - 1} = \frac{x^2}{2x - 1} \). By the second equation this is also \( y + \Delta y \). Now find the basin:

If \( x < 0 \) then \( \Delta x > 0 \) but \( x + \Delta x \) still < 0: moving toward 0. If \( 0 \leq x < \frac{1}{2} \) then \( x \) is \( \Delta x < 0 \). So the basin for \((0,0)\) has all \( x \leq \frac{1}{2} \). The line \( z = \frac{1}{2} \) gives blowup. If \( \frac{1}{2} < x \leq 1 \) then \( \Delta x > 0 \). If \( x > 1 \) then \( x + \Delta x < 0 \) but \( x + \Delta x = \frac{x^2}{2x - 1} \) \( \geq 1 \) (because \( x^2 - 2x + 1 \geq 0 \)). So the basin for \((1,1)\) has all \( x > \frac{1}{2} \).
42 $J = \begin{bmatrix} 2 & 2 \\ 1 & 1 \end{bmatrix}$ is singular; $g$ and $h$ have the same tangent planes. Newton’s equations $2\Delta x + 2\Delta y = -2$ and $\Delta x + \Delta y = -1$ have infinitely many solutions.

13.4 Directional Derivatives and Gradients (page 495)

$D_u f$ gives the rate of change of $f(x,y)$ in the direction $u$. It can be computed from the two derivatives $\partial f/\partial x$ and $\partial f/\partial y$ in the special directions $(1,0)$ and $(0,1)$. In terms of $u_1, u_2$ the formula is $D_u f = f_{x} u_1 + f_{y} u_2$. This is a dot product of $u$ with the vector $(f_x, f_y)$, which is called the gradient. For the linear function $f = ax + by$, the gradient is $\text{grad } f = (a, b)$ and the directional derivative is $D_u f = (a, b) \cdot u$.

The gradient $\nabla f = (f_x, f_y)$ is not a vector in three dimensions, it is a vector in the base plane. It is perpendicular to the level lines. It points in the direction of steepest climb. Its magnitude $|\text{grad } f|$ is the steepness $\sqrt{f_x^2 + f_y^2}$. For $f = x^2 + y^2$ the gradient points out from the origin and the slope in that steepest direction is $|D_u f| = 2r$.

The gradient of $f(x,y,z)$ is $(f_x, f_y, f_z)$. This is different from the gradient on the surface $F(x,y,z) = 0$, which is $-(F_x/F_z)i - (F_y/F_z)j$. Traveling with velocity $v$ on a curved path, the rate of change of $f$ is $df/dt = (\text{grad } f) \cdot v$. When the tangent direction is $T$, the slope of $f$ is $df/ds = (\text{grad } f) \cdot T$. In a straight direction $u$, $df/ds$ is the same as the directional derivative $D_u f$.
13.4 Directional Derivatives and Gradients (page 495)

13.4 Directional Derivatives and Gradients

45 \text{v} = (2, 3); \text{T} = \frac{\text{v}}{\text{v}}; \frac{df}{dx} = \text{v} \cdot (2x_0 + 4t, -2y_0 - 6t) = 4x_0 - 6y_0 - 10t; \frac{df}{dt} = \frac{df}{dt} = \frac{1}{\sqrt{15}}

46 \text{v} = (e^t, 2e^{-t}, -e^t); \text{T} = \frac{\text{v}}{\text{v}}; \text{grad} \ f = (\frac{1}{2}, \frac{1}{2}) = (e^t, -e^{-2t}, e^t), \frac{df}{dx} = 1 + 2t - 1, \frac{df}{dx} = \frac{2}{\sqrt{15}}

47 \text{v} = (-2 \sin 2t, 2 \cos 2t), \text{T} = (-2 \sin 2t, \cos 2t); \text{grad} \ f = (y, x), \frac{df}{dx} = -2 \sin 2t + 2 \cos 2t, \frac{df}{dx} = \frac{1}{2}

zero slope because \ f = 1 on this path

49 \text{v} = 2(x - 4) + 3(y - 5); \text{T} = 1 + 2(x - 4) + 3(y - 5) \quad \text{grad} \ f \cdot \text{T} = 0; \text{T}

51 \text{grad} \ f = \text{T} = 0; \text{T}

2 \text{grad} \ f = f_x + f_y = 3 \text{al} + 4 \text{al}; \text{D}_u f = 3(\frac{\text{al}}{\text{al}}) + 4(\frac{\text{al}}{\text{al}}) = 5 \text{ at every point } \text{P}.

4 \text{grad} \ f = 10y^2; \text{D}_u f = -10y^2; \text{D}_u f(P) = 10. \quad \text{grad} \ f = \frac{1}{x^2 + y^2 + z^2} \quad \text{which makes the slope equal to} \frac{3 \text{al}}{10} \quad \text{above the line} \text{y} = 2x \text{ and } \text{f} = y \text{ below that line. \quad The two pieces agree on the line. Then \quad grad} \ f = 2 \text{al above and grad} \ f = \text{j below. Surprisingly \ f increases fastest along the line, which is the direction } \text{u} = \frac{1}{x^2 + y^2 + z^2} \quad \text{and gives} \ D_u f = \frac{2 \text{al}}{5}.

14 \text{grad} \ f = \frac{1}{x^2 + y^2 + z^2} \quad \text{and } \text{P} \text{ is a rough point! The rate of increase is infinite (provided } x^2 + y^2 \text{ stays below 5; the direction must point into this circle).}

18 \text{N} \cdot \text{U} = \text{N} \cdot \text{L} = \text{U} \cdot \text{L} = 0 \quad \text{(b) \text{N} is perpendicular to the tangent plane, \text{U} and \text{L are parallel to the tangent plane. (c) \ The gradient is the xy projection of \text{N} and also of \text{U. The projection of \text{L points along the level curve.}}}

20 \text{N} = (\frac{x}{\sqrt{x^2 + y^2}}, \frac{y}{\sqrt{x^2 + y^2}}, -1), \text{U} = (\frac{x}{\sqrt{x^2 + y^2}}, \frac{y}{\sqrt{x^2 + y^2}}, 1), \text{and } \text{L} = (\frac{x}{\sqrt{x^2 + y^2}}, \frac{-y}{\sqrt{x^2 + y^2}}, 0). \text{U goes up the side of the cone.}

22 \text{U} = (-4, 3, -5). \text{The xy direction of flow is } - \text{grad z} = -4 \text{i + 3j}.

24 \text{U} = (\frac{-x}{\sqrt{x^2 + y^2}}, \frac{-y}{\sqrt{x^2 + y^2}}, -1). \text{The xy direction of flow is radially inward.}

26 \frac{x}{y} = \frac{1}{1} \text{ is a straight level curve } y = x. \text{The direction of the gradient is perpendicular to that level curve:}

\text{gradient along } -\text{i + j}. \text{Check: grad} \ f = \frac{-y}{x} + \frac{1}{x} = -1 + j.

28 \text{(a) False because } f + C \text{ has the same gradient as } f \text{ (b) True because the line direction } (1, 1, -1) \text{ is also the normal direction N (c) False because the gradient is in 2 dimensions.}

30 \theta = \tan^{-1} \frac{y}{x} \text{ has grad } \theta = (\frac{-y}{1 + (y/x)} \frac{x}{1 + (y/x)}) = (\frac{-y}{x^2 + y^2}, \frac{x}{x^2 + y^2}). \text{The unit vector in this direction is}

\text{T} = (\frac{x}{x^2 + y^2}, \frac{y}{x^2 + y^2}). \text{Then grad } \theta \cdot \text{T} = (\frac{x^2 + y^2}{x^2 + y^2})^2 = \frac{1}{x^2 + y^2}

32 \text{T} = e^{-x^2 - y^2} \text{ has } \Delta T = \frac{\partial T}{\partial x} \Delta x + \frac{\partial T}{\partial y} \Delta y = -(2x \Delta x - 2y \Delta y) e^{-x^2 - y^2} = (-2 \Delta x + 4 \Delta y) e^{-5}. \text{This is largest going in toward } (0,0), \text{in the direction } u = (-\frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}}).

34 \text{The gradient is } (2ax + c \text{i + (2by + d)j}. \text{The figure shows } c = 0 \text{ and } d \approx \frac{1}{3} \text{ at the origin. Then } b \approx \frac{1}{3} \text{ from the gradient at } (0,1). \text{Then } a \approx -\frac{1}{4} \text{ from the gradient at } (2,0). \text{The function } -\frac{1}{4} x^2 + \frac{1}{3} y^2 + \frac{1}{3} y \text{ has hyperbolas opening upwards as level curves.}

36 \text{grad} \ f \text{ is tangent to } xy = c \text{ and therefore perpendicular to } yl + xj. \text{So grad } f \text{ is a multiple of } xj - yj.

\text{|grad} \ f \text{ is larger at } Q \text{ than } P. \text{It is not constant on the hyperbolas. The function could be } f = x^2 - y^2.

\text{Its level curves are also hyperbolas, perpendicular to those in the figure.}

38 \text{f(0,1) = B + C = 0, } f(1,0) = A + C = 1, \text{ and } f(2,1) = 2A + B + C = 2. \text{Solution } A = 1, B = C = 0.

So grad } f = 1.
The Chai. Rule (page 503)

40 The function is \( xy + C \) so its level curves are standard hyperbolas.

42 \( \mathbf{v} = (\frac{dx}{dt}, \frac{dy}{dt}) = (-2 \sin 2t, 2 \cos 2t); \mathbf{T} = (-\sin 2t, \cos 2t) \); \( \text{grad} \, f = (1, 0) \) so \( \frac{dy}{dx} = -2 \sin 2t \) and \( \frac{df}{ds} = \sin 2t \).

44 \( \mathbf{v} = (2t, 0) \) and \( \mathbf{T} = (1, 0) \); \( \text{grad} \, f = (x, z) \) so \( \frac{df}{ds} = y = 3 \).

46 \( \mathbf{v} = (1, 2t, 3t^2) \) and \( \mathbf{T} = (\sin t, \cos t) \); \( \text{grad} \, f = (4x, 6y, 2z) \) so \( \frac{df}{ds} = 4t + 12t^3 + 6t^5 \) and \( \frac{df}{ds} = \sqrt{1 + 4t^2 + 9t^4} \).

48 \( D = (x - 1)^2 + (y - 2)^2 \) has \( \frac{df}{dy} = 2(z - 1) \) or \( \frac{df}{dx} = \frac{2}{D} \). Similarly \( 2D \frac{dy}{dy} = 2(y - 2) \) and \( \frac{df}{dx} = \frac{2}{D} \).

Then \( \text{grad} \, f = (\frac{df}{dx}, \frac{df}{dy}) \) so \( \Delta f = \frac{df}{dx} \Delta x + \frac{df}{dy} \Delta y = D_{\Delta x} f(P) \) times \( \Delta x \) and \( D_{\Delta y} f(P) = u_1 \frac{df}{dx}(P) + u_2 \frac{df}{dy}(P) \).

13.5 The Chain Rule (page 503)

The chain rule applies to a function of a function. The \( x \) derivative of \( f(g(x, y)) \) is \( \partial f / \partial x = (\partial f / \partial g)(\partial g / \partial x) \). The \( y \) derivative is \( \partial f / \partial y = (\partial f / \partial g)(\partial g / \partial y) \). The example \( f = (x + y)^n \) has \( g = x + y \). Because \( \partial g / \partial x = \partial g / \partial y \) we know that \( \partial f / \partial x = \partial f / \partial y \). This partial differential equation is satisfied by any function of \( x + y \).

Along a path, the derivative of \( f(x(t), y(t)) \) is \( \frac{df}{dt} = (\partial f / \partial x)(dx/dt) + (\partial f / \partial y)(dy/dt) \). The derivative of \( f(x(t), y(t), z(t)) \) is \( fx(t) + fyyt \). If \( f = xy \) then the chain rule gives \( \frac{df}{dt} = y dx/dt + x dy/dt \). That is the same as the product rule! When \( x = u_1t \) and \( y = u_2t \) the path is a straight line. The chain rule for \( f(x, y) \) gives \( \frac{df}{dt} = u_1 \frac{df}{dx}(P) + u_2 \frac{df}{dy}(P) \).

The chain rule for \( (z(t, u), y(t, u)) \) is \( \partial f / \partial t = (\partial f / \partial x)(\partial x / \partial t) + (\partial f / \partial y)(\partial y / \partial t) \). We don't write \( \partial f / \partial t \) because \( f \) also depends on \( u \). If \( x = r \cos \theta \) and \( y = r \sin \theta \), the variables \( t, u \) change to \( r \) and \( \theta \). In this case \( \partial f / \partial r = (\partial f / \partial x) \cos \theta + (\partial f / \partial y) \sin \theta \) and \( \partial f / \partial \theta = (\partial f / \partial x)(-r \sin \theta) + (\partial f / \partial y)(r \cos \theta) \). That connects the derivatives in rectangular and polar coordinates. The difference between \( \partial r / \partial x = x/r \) and \( \partial r / \partial y = 1/cos \theta \) is because \( y \) is constant in the first and \( \theta \) is constant in the second.

With a relation like \( xyz = 1 \), the three variables are not independent. The derivatives \( (\partial f / \partial x)_z \) and \( (\partial f / \partial y)_x \) mean that \( y \) is held constant, and \( z \) is constant, and both are constant. For \( f = x^2 + y^2 + z^2 \) with \( xyz = 1 \), we compute \( (\partial f / \partial x)_z \) from the chain rule \( \partial f / \partial x + (\partial f / \partial y)(\partial y / \partial x) \). In that rule \( \partial z / \partial x = -1/x^2y \) from the relation \( xyz = 1 \).

\[
\begin{align*}
1 \quad f_x &= f_y = \cos(x + y) & \quad 3 \quad f_y &= c f_x = c \cos(x + cy) & \quad 5 \quad 3g^2 \frac{dy}{dx} \frac{ds}{dt} + 3g^2 \frac{dy}{dy} \frac{du}{dt} & \quad 7 \quad \text{Moves left at speed 2} \\
9 \quad \frac{dx}{dt} &= 1 \quad \text{(wave moves at speed 1)} & \quad 11 \quad \frac{\partial^2}{\partial x^2} f(x + iy) &= f''(x + iy), \frac{\partial^2}{\partial y^2} f(x + iy) = i^2 f''(x + iy) & \quad \text{so } f_{xx} + f_{yy} = 0; (x + iy)^2 = (x^2 - y^2) + i(2xy) \\
13 \quad \frac{d}{dt} &= 2x(1) + 2y(2t) = 2t + 4t^3 & \quad 15 \quad \frac{df}{dt} &= y \frac{ds}{dt} + z \frac{du}{dt} = -1 & \quad 17 \quad \frac{df}{dt} = \frac{1}{x+y} \frac{dx}{dt} + \frac{1}{x+y} \frac{dy}{dt} = 1 \\
19 \quad V = \frac{1}{2}r^2 \rho \frac{\partial \rho}{\partial r} \frac{dr}{dt} + \frac{2 \rho r \frac{\partial \rho}{\partial r}}{3} + \frac{\rho^2 \frac{\partial \rho}{\partial r}}{3} = 8 \pi \\
21 \quad \frac{dD}{dt} = \frac{90}{\sqrt{90^2 + 90^2}} (60) + \frac{90}{\sqrt{90^2 + 90^2}} (45) = \frac{105}{2} \text{ mph}; \quad \frac{dD}{dt} = \frac{90}{\sqrt{45^2 + 60^2}} (60) + \frac{45}{\sqrt{45^2 + 60^2}} (45) = 74 \text{ mph}
\end{align*}
\]
23 $\frac{df}{dt} = u \frac{df}{du} + v \frac{df}{dv} + w \frac{df}{dw}$ 25 $\frac{df}{dt} = 1$ with $x$ and $y$ fixed; $\frac{df}{dt} = 6$

27 $f_t = f_x t + f_y (2t); f_{tt} = f_{xx} t + f_y (2t) + 2f_{yt} = (f_{xx} t + f_y (2t)) t + f_x + 2(f_{xy} t + f_{yy} (2t)) t + 2f_y$

29 $\frac{df}{dt} = \frac{df}{dx} \frac{dx}{dt} + \frac{df}{dy} \frac{dy}{dt} = \frac{df}{dx} \cos \theta + \frac{df}{dy} \sin \theta$, $\theta$ is fixed

31 $r_{xx} = \frac{1}{\sqrt{x^2+y^2}} - \left(\frac{x^2+y^2}{x^2+y^2}\right) \frac{\partial^2}{\partial x^2} \left(\frac{1}{\sqrt{x^2+y^2}}\right)$; $\frac{\partial}{\partial x} \left(\frac{1}{x} - \frac{x^2}{x^2+y^2}\right) = -\frac{x^2}{x^2+y^2}$

33 $\left(\frac{1}{r^2}\right) v = \frac{1}{\sqrt{1-\cos^2 x}}$; first answer is also $\frac{1}{\sqrt{1-\cos^2 x}}$

35 $f_t = f_x \cos \theta + f_y \sin \theta, f_{tt} = f_{xx} \cos \theta + f_{xy} \sin \theta - f_x \sin \theta + f_y \cos \theta = f_x \sin \theta + f_y \cos \theta + f_{xx} \sin \theta \cos \theta + f_{xy} \sin \theta \cos \theta + f_{xy} \cos \theta \cos \theta + f_{yy} \cos \theta \cos \theta$

37 Yes (with $y$ constant): $\frac{\partial}{\partial y} = y e^{x^2}, \frac{\partial}{\partial z} = \frac{\partial}{\partial z} = \frac{1}{y e^{x^2}}$ 39 $f_t = f_x + f_y; f_{tt} = f_{xx} + 2f_{xy} + f_{yy}$

41 $\left(\frac{1}{r^2}\right) x = \frac{r}{\partial x} + \frac{r}{\partial y} = -\frac{a}{b} + \frac{b}{a} \frac{\partial}{\partial x} = -\frac{a}{b} \frac{\partial}{\partial y}$

43 $1$ 45 $f = y^2$ so $f_t = 0, f_{yy} = 2y = 2r \sin \theta$; $f = r^2$ so $f_t = 2r = 2 \sqrt{x^2+y^2}$, $f_{xx} = 0$

47 $g_u = f_x x_t + f_y y_t = f_x + f_y; g_{yy} = f_x x_t + f_y y_t = f_x - f_y; g_{uu} = f_x x_u + f_y y_u + f_x x_u + f_y y_u$

$= f_{xx} + 2f_{xy} + f_{yy}; g_{uu} = f_{xx} x_x + f_{yy} y_y - f_{xx} x_x - f_{yy} y_y = f_{xx} - 2f_{xy} + f_{yy}$. Add $g_{uu} + g_{uv}$ 49 True

2 $f_x = 10a (ax + by)^0$ and $f_y = 10b (ax + by)^0; f_{xx} = af_x$. 4 $f_x = \frac{1}{x+y}; f_y = \frac{1}{x-y}; f_{xx} = f_y$

6 $\frac{df}{dt} = \frac{df}{dx} \frac{dx}{dt} + \frac{df}{dy} \frac{dy}{dt}$ is the product rule $\frac{y}{\sqrt{2\frac{\pi}{\sqrt{2}}} + x \frac{\sqrt{2}}{\sqrt{2}}}$. In terms of $u$ and $v$ this is $\frac{df}{dt} = \frac{v}{\sqrt{2\frac{\pi}{\sqrt{2}}} + u \frac{\sqrt{2}}{\sqrt{2}}}$.

8 $f_{tt} = c^2 (\sin n-1)(x+ct)^{n-2}$ which equals $c^2 f_{xx}$. Choose $C = -c: f = (x - ct)$ also has $f_{tt} = c^2 f_{xx}$.

10 Since $\sin (0 - t)$ is decreasing (it is $-\sin t$), you go down. At $t = 4$, your height is $-\sin 4$ and your velocity is $- \cos 4 = -\cos 4$.

12 (a) $f_x = 2 e^{i \theta} f_r, f_y = 2 e^{i \theta} f_r, f_{rr} = (2i)^2 e^{i \theta}$ and $f_{rr} = \frac{e^{i \theta}}{r^2} f_{rr}$ and $f_{rr} = f_r^{(i \theta)} f_r^{(i \theta)}$ and $f_{rr} = f_r^{(i \theta)} f_r^{(i \theta)}$ and $f_{rr} = f_r^{(i \theta)} f_r^{(i \theta)}$.

Any $f(r \theta)$ or any $f(x + iy)$ will satisfy the polar or rectangular form of Laplace's equation.

14 $\frac{df}{dt} = \frac{1}{\sqrt{2\frac{\pi}{\sqrt{2}}} + x \frac{\sqrt{2}}{\sqrt{2}}} (1 + \sqrt{\frac{x}{\sqrt{2}}} + x \frac{\sqrt{2}}{\sqrt{2}}) = \frac{1 + \sqrt{x} + x \frac{\sqrt{2}}{\sqrt{2}}}{\sqrt{2\frac{\pi}{\sqrt{2}}} + x \frac{\sqrt{2}}{\sqrt{2}}}.$

16 Since $x = \frac{1}{2}$ we must find $\frac{df}{dt} = 0$. The chain rule gives $\frac{\partial}{\partial x} - \frac{\partial}{\partial y} = \frac{1}{2} (x^2) - \frac{1}{4} (2x^3) = 0$.

18 $\frac{df}{dt} = (4x^2)(1) + (0)(1) = 4x^2$

20 The rocket's position is $x = 6t, y = t^2$. Its speed from $(0,0)$ is $\frac{dx}{dt} = 6, \frac{dy}{dt} = 2t$. At $t = 0$ this speed is $\frac{\partial}{\partial \theta} \frac{\theta}{1} = 6$. The rate of change of $\theta = \tan^{-1} t$ is $\frac{\partial}{\partial \theta} \frac{\theta}{1} = \frac{1}{1+\frac{1}{2} t^2}$. At $t = 0$ this is $\frac{1}{2}$.

22 Driving south $\frac{df}{dt} = (0.05)(70) = 3.5$ degrees per hour. Southeast now gives $\frac{df}{dt} = (0.05) \frac{80}{\sqrt{2}} + (0.01) \frac{80}{\sqrt{2}}$.

$\approx 3.4$ degrees per hour. $\frac{df}{dt}$ is larger going south.

$\frac{df}{dt} = \frac{df}{dx} \frac{dx}{dt} + \frac{df}{dy} \frac{dy}{dt} = (f_x u_1 + f_y u_2 + f_z u_3) u_1 + (f_x u_1 + f_y u_2 + f_z u_3) u_2 + (f_x u_1 + f_y u_2 + f_z u_3) u_3 = f_x u_1^2 + f_y u_2^2 + f_z u_3^2$. For $f = x y z$ this is $2x u_1 + 2y u_2 + 2z u_3$.

$\frac{df}{dt} = 0.90 (ax + by + c)^8$ and $(at + bt + c)^9$ has $f_{tt} = 90(a + b)^2 (at + bt + c)^8$. It is false that $\frac{df}{dt} = 0.9$ (we also need the term $\frac{df}{dy} \frac{dy}{dt}$).

32 $\frac{df}{dy} = \frac{\partial f}{\partial y} \frac{dy}{dt}$ and then $\frac{\partial f}{\partial x} \frac{dx}{dt} = -\frac{\partial f}{\partial x} \frac{dy}{dt} = -\frac{\partial f}{\partial y} \frac{dx}{dt}$.

$\frac{df}{dt} = \frac{\partial f}{\partial x} \frac{dx}{dt} + \frac{\partial f}{\partial y} \frac{dy}{dt} + \frac{\partial f}{\partial z} \frac{dz}{dt} = \frac{\partial f}{\partial x} \frac{dx}{dt} + \frac{\partial f}{\partial y} \frac{dy}{dt} + \frac{\partial f}{\partial z} \frac{dz}{dt}$.

36 Yes, if $y$ is simply held constant then the old rule continues to apply.
38 \( \frac{\partial^2 f}{\partial x \partial y} (z, y, z) = \frac{\partial^2 f}{\partial z \partial x} + \frac{\partial^2 f}{\partial z \partial y} + \frac{\partial^2 f}{\partial x \partial y} \).

40 (a) \( \frac{\partial f}{\partial x} = 2x \) (b) \( f = x^2 + y^2 + (z^2 + y^2)^2 \) so \( \frac{\partial f}{\partial x} = 2x + 4x(x^2 + y^2) \) (d) \( y \) is constant for \( \frac{\partial f}{\partial x} \).

42 (\( \frac{\partial^2 f}{\partial x \partial y} \)) = -\( \frac{\partial f}{\partial x} \) and similarly \( \frac{\partial^2 f}{\partial y \partial x} \). Multiply these three equations: the right hand sides produce \(-1\).

44 \( \frac{\partial f}{\partial x} = \frac{\partial f}{\partial x} (1) + \frac{\partial f}{\partial y} (u) \) and \( \frac{\partial f}{\partial y} = \frac{\partial f}{\partial y} (1) + \frac{\partial f}{\partial u} (t) \). For \( f = x^2 - 2y \) these become \( \frac{\partial f}{\partial x} = 2x(1) - 2(t + u) = 2(t + u) - 2u = 2t \) and similarly \( \frac{\partial f}{\partial u} = 2u \).

46 \( \sin x + \sin y = 0 \) gives \( \cos x = \cos y = 0 \) and \(-\sin x - \sin y \neq 2 + \sin y \) then \( \frac{\partial f}{\partial x} = -\frac{\cos x}{\cos y} \) and \( \frac{\partial^2 f}{\partial x^2} = \frac{\sin x + \sin y \cos^2 x}{\cos^2 y} \).

48 The \( \tau \) derivative of \( f(x, y) = \cos x + \cos y \) is \( x \frac{\partial f}{\partial x} (x, y) + y \frac{\partial f}{\partial y} (x, y) = f(x, y) \). At \( \tau = 1 \) this becomes \( x \frac{\partial f}{\partial x} (x, y) + y \frac{\partial f}{\partial y} (x, y) = f(x, y) \). Test on \( f = \sqrt{x^2 + y^2} : x \frac{\partial f}{\partial x} = \frac{x}{\sqrt{x^2 + y^2}} \) and \( y \frac{\partial f}{\partial y} = \frac{y}{\sqrt{x^2 + y^2}} \) = \( \sqrt{x^2 + y^2} \). Other examples: \( f(x, y) = \sqrt{x^2 + 2xy + y^2} \) or \( f = Ax + By \) or \( f = x^{1/4}y^{3/4} \).

13.6 Maxima, Minima, and Saddle Points (page 512)

A minimum occurs at a stationary point (where \( f_x = f_y = 0 \)) or a rough point (no derivative) or a boundary point. Since \( f = x^2 - xy + 2y \) has \( f_x = 2x - y \) and \( f_y = 2 - x \), the stationary point is \( x = 2, y = 4 \). This is not a minimum, because \( f \) decreases when \( y \) increases.

The minimum of \( d^2 = (x - x_1)^2 + (y - y_1)^2 \) occurs at the rough point \( (x_1, y_1) \). The graph of \( d \) is a cone and \( \nabla d \) is a unit vector that points out from \( (x_1, y_1) \). The graph of \( f = lxy \) touches bottom along the lines \( x = 0 \) and \( y = 0 \). Those are "rough lines" because the derivative does not exist. The maximum of \( d \) and \( f \) must occur on the boundary of the allowed region because it doesn't occur inside.

When the boundary curve is \( x = x(t), y = y(t) \) the derivative of \( f(x, y) \) along the boundary is \( f_x x_t + f_y y_t \) (chain rule). If \( f = x^2 + 2y^2 \) and the boundary is \( x = \cos t, y = \sin t \), then \( df/dt = 2 \sin t \cos t \). It is zero at the points \( t = 0, \pi/2, \pi, 3\pi/2 \). The maximum is at \( (0, \pm 1) \) and the minimum is at \( (\pm 1, 0) \). Inside the circle \( f \) has an absolute minimum at \( (0, 0) \).

To separate maximum from minimum from saddle point, compute the second derivatives at a stationary point. The tests for a minimum are \( f_{xx} > 0 \) and \( f_{xy} f_{yy} > f_{yx}^2 \). The tests for a maximum are \( f_{xx} < 0 \) and \( f_{xy} f_{yy} > f_{yx}^2 \). In case \( ac < b^2 \) or \( f_{xx} f_{yy} < f_{xy}^2 \), we have a saddle point. At all points these tests decide between concave up and concave down and "indefinite". For \( f = 8x^2 - 6xy + y^2 \), the origin is a saddle point. The signs of \( f \) at \( (1, 0) \) and \( (1, 3) \) are + and -.

The Taylor series for \( f(x, y) \) begins with the terms \( f(0, 0) + xf_x + yf_y + \frac{1}{2}x^2f_{xx} + xyf_{xy} + \frac{1}{2}y^2f_{yy} \). The coefficient of \( x^n y^m \) is \( \frac{\partial^n f}{\partial x^n \partial y^m} (0, 0) \) divided by \( n!m! \) To find a stationary point numerically, use New-
ton’s method or steepest descent.

1. (0, 0) is a minimum.
2. (3, 0) is a saddle point.
3. (0, 3) is a saddle point.
4. (0, 0) is a saddle point.
5. No stationary points.
6. (7, 0) is a maximum.
7. (9, 0, 2) is a minimum.
8. (11, 0) is a minimum on the line x = y are minima.
9. (13, 0, 0) is a saddle point.
10. (15, 0, 0) is a saddle point; (2, 0) is a minimum; (0, -2) is a maximum; (2, -2) is a saddle point.

17. Maximum of area \((12 - 3y)y\) is 12.
\[
2(x + y) + 2(x + 2y - 5) + 2(x + 3y - 4) = 0 \quad \text{gives} \quad x = 2; y = -1 \quad \text{min because} \quad E_{xx}E_{yy} = (6)(28) > E_{x}^{2} = 12^{2}
\]

21. Minimum at \((0, \frac{3}{2}); (0, 1); (0, 1)\)
\[
\frac{d^{2}y}{dx} = 0 \quad \text{when} \quad \tan t = \sqrt{3}; f_{\max} = 2 \quad \text{at} \quad (\frac{1}{2}, \sqrt{3}) \quad f_{\min} = -2 \quad \text{at} \quad (-\frac{1}{2}, -\sqrt{3})
\]

25. \(ax + by)_{\text{max}} = \sqrt{a^{2} + b^{2}; (x^{2} + y^{2})_{\text{min}} = \frac{1}{a^{2} + b^{2}}}
\]

27. 0 < c < \frac{1}{4}

29. The vectors head-to-tail form a 60-60-60 triangle. The outer angle is 120°.
31. 2 + \sqrt{3}; 1 + \sqrt{3}; 1 + \sqrt{5}

50. Steiner point where the arcs meet.
39. Best point for \(p = \infty\) is equidistant from corners.

41. Grad \(f = (\sqrt{2} - \frac{z}{d_{1}}) + \frac{z}{d_{2}} + \frac{z}{d_{3}} \sqrt{\frac{1}{d_{1}} + \frac{1}{d_{2}} + \frac{1}{d_{3}}}; \text{angles are} \quad 90-135-135\) 39. Best point for \(p = \infty\) is equidistant from corners.

43. Third derivatives all 6; \(f = \frac{8}{3} x^{2} + \frac{8}{3} x^{2}y + \frac{8}{3} xy^{2} + \frac{8}{3} y^{3}\)

46. \((\frac{2}{dy} + (\frac{2}{dy})^{m})_{n} \quad \text{ln}(1-sy)_{n}= m(n-1)\) for \(m = n > 0\), other derivatives zero; \(f = -x^{2} - \frac{y^{2}}{2} - \frac{x^{2}}{2} - \ldots\)

47. All derivatives are \(c^{2} \text{ at} \quad (1, 1); f \approx c^{2}[(x - 1) + (y - 1) + \frac{1}{2}(x - 1)^{2} + (x - 1)(y - 1) + \frac{1}{2}(y - 1)^{2}]

49. \(x = 1, y = -1 \quad f_{x} = 2, f_{y} = -2, f_{xx} = 2, f_{xy} = 0, f_{yy} = 2 ; \text{series must recover} \quad x^{2} + y^{2}\)

51. Line \(x - 2y = \text{constant} \quad x + y = \text{constant}\)

52. \(\frac{3}{2} f_{xx} + xyf_{y} + \frac{3}{2} f_{yy} = 0 \quad \text{and} \quad f_{xx} f_{yy} > f_{xy}^{2}\) at \((0, 0); f_{x} = f_{y} = 0\)

55. \(\Delta x = -1, \Delta y = 1\)

57. \(f = x^{2}(12 - 4z)\) has \(f_{\max} = 16 \quad \text{at} \quad (2, 4); \text{line has slope} \quad -4, y = \frac{13}{2}\) has slope \(\frac{13}{2} = -4\)

59. If the fence were not perpendicular, a point to the left or right would be closer.

\(\begin{align*}
2 f_{x} = y - 1, f_{y} = x - 1; b^{2} - ac = 1; (1, 1) \text{ is a saddle point.} \\
4 f_{x} = 2x, f_{y} = -2y + 4; b^{2} - ac = 1; (0, 2) \text{ is a saddle point.} \\
6 f_{x} = e^{x} - e^{y}, f_{y} = xe^{y}; (0, 0) \text{ is the stationary point; } f_{xx} = -e^{y} = -1, f_{xy} = e^{y} = 1, f_{yy} = xe^{y} = 0 \text{ so } b^{2} - ac = 1 \text{: saddle point.} \\
8 f_{x} = 2(x + y) + 2(x + 2y - 6), f_{y} = 2(x + y) + 4(x + 2y - 6); (-6, 6) \text{ is the stationary point: } f_{xx} = 4, f_{xy} = 6, f_{yy} = 10 \text{ give } b^{2} - ac = -4 \text{: minimum.} \\
10 f_{x} = x + 2y - 6 + x + y \text{ and } f_{y} = x + 2y - 6 + 2(x + y); (-6, 6) \text{ is the stationary point; } f_{xx} = 2, f_{xy} = 3, f_{yy} = 4 \text{ give } b^{2} - ac = 9 - 8 = 1 \text{: saddle point.} \\
12 f_{x} = \frac{2x}{1+y^{2}} \text{ and } f_{y} = x(1+x+y^{2}); (0, 0) \text{ is the stationary point; } f_{xx} = \frac{2}{1+y^{2}} = 2, f_{xy} = \frac{-4x}{(1+y^{2})^{2}} = 0, f_{yy} = \frac{-2(1+y^{2})}{(1+y^{2})^{2}} \text{ is a saddle point.} \\
14 f_{x} = \cos x \text{ and } f_{y} = \sin y; \text{ stationary points have } x = \frac{\pi}{2} + n\pi \text{ and } y = m\pi; \text{ maximum when } f = 2, \text{ saddle point when } f = 0, \text{ minimum when } f = -2. \\
16 f_{x} = 8y - 4z, f_{y} = 8z - 4y^{3}; \text{ stationary points are } (0, 0) \text{ is saddle point. } (\sqrt{2}, \sqrt{2}) \text{ is maximum, } (\sqrt{2}, -\sqrt{2}) \text{ is minimum.} \\
18 \text{Volume } = xyz = zy(1 - 3x - 2y) = xy - 3x^{2} - 2xy^{2}; V_{y} = y - 6x - 2y^{2} \text{ and } V_{y} = x - 4xy; \text{ at } (0, \frac{1}{3}, 0) \text{ and } (\frac{1}{3}, 0, 0) \text{ and } (0, 0, 1) \text{ the volume is } V = 0 \text{ (minimum)}; \text{ at } (\frac{1}{48}, \frac{13}{48}, \frac{21}{48}) \text{ the volume is } V = \frac{7}{3072} \text{ (maximum).} \\
20 \text{Minimise } f(x, y) = (x - y + 1)^{2} + (x + y + 1)^{2} + (x + y - 1)^{2} + 2(x + y + 1) + 4(x + y - 1) + 0 \text{ and } \frac{\partial f}{\partial x} = (x - y - 1) + 2(2x + y + 1) + 4(x + y - 1) = 0. \text{ Solution: } x = y = 0! \\
22 \frac{\partial f}{\partial x} = 2x + 2 \text{ and } \frac{\partial f}{\partial y} = 2y + 4. \text{ (a) Stationary point } (-1, -2) \text{ yields } f_{min} = -5. \text{ (b) On the boundary } y = 0 \text{ the minimum of } x^{2} + 2x \text{ is } -1 \text{ at } (-1, 0) \text{ (c) On the boundary } x \geq 0, y \geq 0 \text{ the minimum is } 0 \text{ at } (0, 0).\)
13.7 Constraints and Lagrange Multipliers (page 519)

24 \( f\left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right) = \frac{5}{2} - \sqrt{2}; f\left(-\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}\right) = \frac{5}{2} + \sqrt{2} = f_{\text{max}} \); \( f(1,0) = f(0,1) = 1 = f_{\text{min}} \)

26 \( f_x = x^2 - y = 0 \) and \( f_y = y^2 - z = 0 \) combine into \( y = x^2 = y^2 \). Then \( y = y^2 \) gives \( y = 1 \) or \(-1 \) or 0.

At those points \( f_{\text{min}} = -\frac{1}{2} \) and \( f = 0 \) (relative maximum). These equations \( x^2 = y, y^2 = z \) are solved by Newton's method in Section 13.3 (the basins are on the front cover).

28 \( d_1 = x, d_2 = d_3 = \sqrt{(1-x^2)^2 + 1}, d_4 = (x+2\sqrt{(1-x^2)^2 + 1}) = 1 + \frac{2(1-x^2)}{\sqrt{(1-x^2)^2 + 1}} = 0 \) when \((1-x^2)^2 + 1 = 4(x-1)^2\)

or \( 1-x = \frac{1}{\sqrt{3}} \) or \( z = 1 - \frac{1}{\sqrt{3}} \). From that point to \((1,1)\) the line goes up 1 and across \( \frac{1}{\sqrt{3}} \), a 60° angle with the horizontal that confirms three 120° angles.

30 \( d_1 = \sqrt{(x-x_1)^2 + (y-y_1)^2 + (z-z_1)^2} \) and then \( \frac{d_1}{dx} = x + 2\sqrt{(1-x^2)^2 + 1} \) is a unit vector. The sum is solved by Newton's method in Section 13.3 (the basins are on the front cover).

At those points \( f_{\text{min}} = -\frac{1}{2} \) and \( f = 0 \) (relative maximum). These equations \( x^2 = y, y^2 = z \) are solved by Newton's method in Section 13.3 (the basins are on the front cover).

32 From an outside point the lines to the three vertices give two angles that add to less than 180°. So they cannot both be 120° as a Steiner point requires.

34 From the point \( C = (0,-\sqrt{3}) \) the lines to \((-1,0)\) and \((0,0)\) make a 60° angle. \( C \) is the center of the circle \( x^2 + (y-\sqrt{3})^2 = 4 \) through those two points. From any point on that circle, the lines to \((-1,0)\) and \((0,0)\) make an angle of 2 \( \times 60° = 120° \). Theorem from geometry: angle from circle = \( 2 \times \) angle from center.

40 The vertices are \((0,0), (1,0), \) and \((0,1)\). The point \((\frac{1}{2}, \frac{1}{2})\) is an equal distance from \((1,-1,-1), \frac{1}{\sqrt{3}}(-1,1,-1), \frac{1}{\sqrt{3}}(1,1,1)\).

The equal angles have \( \cos \theta = -\frac{1}{3} \) by Problem 45 of Section 11.1.

42 For two points, \( d_1 + d_2 \) is a minimum at all points on the line between them. (Note equal 180° angles from the vertices!) For three points, the corner with largest angle is the best corner.

44 \( \frac{\partial}{\partial x} (x^2 y) = x e^y \) for \( n = 0, e^y \) for \( n = 1, \) zero for \( n > 1 \). Taylor series \( x e^y = x + xy + \frac{1}{2}xy^2 + \frac{1}{3}xy^3 + \cdots \)

46 All derivatives equal 1 at \((0,0)\). Quadratic = \( 1 + x + y + \frac{1}{2}x^2 + xy + \frac{1}{3}y^2 \).

48 \( \frac{\partial}{\partial x} (\sin x \cos y) = 1 \) at \((0,0)\) but \( f = f_y = f_{xx} = f_{xy} = 0. \) Quadratic = \( x \).

Check: \( \sin x \cos y \approx (x - \frac{x^3}{6} + \cdots)(1 - \frac{y^2}{2} + \cdots) = x \) to quadratic accuracy.

50 \( f(x + h, y + k) \approx f(x, y) + h \frac{\partial f}{\partial x}(x, y) + k \frac{\partial f}{\partial y}(x, y) + \frac{h^2}{2} \frac{\partial^2 f}{\partial x^2}(x, y) + \frac{h k}{2} \frac{\partial^2 f}{\partial x \partial y}(x, y) + \frac{k^2}{2} \frac{\partial^2 f}{\partial y^2}(x, y) \)

52 \( \frac{\partial}{\partial x} (x \frac{\partial f}{\partial x} + y \frac{\partial f}{\partial y}) = 0 \) at \((0,0,0)\); then \( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} = \frac{\partial^2}{\partial z^2} \) at \((0,0,0)\).

54 \( f = (1-2s)^2 + 2(1-4s)^2 \) has \( \frac{\partial^2}{\partial s^2} = 0 \) at \( s = \frac{3}{10} \). Step ends at \( x = 1 - 2s = \frac{1}{5}, y = 1 - 4s = -\frac{7}{10} \).

56 A maximum has \( f_{zz} < 0 \) and \( f_{yy} < 0 \), so they cannot add to zero. A minimum has \( f_{zz} > 0 \) and \( f_{yy} > 0 \).

The functions \( xy \) and \( z^2 - y^2 \) solve \( f_{xx} + f_{yy} = 0 \) and have saddle points.

58 A house costs \( p, \) a yacht costs \( p^2 \) for \( y \), \( q : \frac{\delta}{\delta z} f(x, \frac{k-2x}{q}) = \frac{\delta}{\delta z} + \frac{\delta}{\delta y} (-\frac{1}{q}) = 0 \) gives \( \frac{\delta}{\delta z} f_{xx} / \frac{\delta}{\delta y} = -\frac{1}{q} \).

13.7 Constraints and Lagrange Multipliers (page 519)

A restriction \( g(x, y) = k \) is called a constraint. The minimizing equations for \( f(x, y) \) subject to \( g = k \) are \( \partial f / \partial x = \lambda \partial g / \partial x, \partial f / \partial y = \lambda \partial g / \partial y, \) and \( g = k. \) The number \( \lambda \) is the Lagrange multiplier. Geometrically, \( f \) is parallel to grad \( g \) at the minimum. That is because the level curve \( f = f_{\text{min}} \) is tangent to the constraint curve \( g = k. \) The number \( \lambda \) turns out to be the derivative of \( f_{\text{min}} \) with respect to \( k. \) The Lagrange function is
13.7 Constraints and Lagrange Multipliers

\[ L = f(x, y) - \lambda (g(x, y) - k) \]

and the three equations for \( x, y, \lambda \) are \( \partial L / \partial x = 0 \) and \( \partial L / \partial y = 0 \) and \( \partial L / \partial \lambda = 0 \).

To minimise \( f = x^2 - y \) subject to \( g = x - y = 0 \), the three equations for \( x, y, \lambda \) are \( 2x = \lambda, -1 = -\lambda, \)

\( x - y = 0 \). The solution is \( x = \frac{1}{2}, y = \frac{1}{2}, \lambda = 1 \). In this example the curve \( f(x, y) = f_{\text{min}} = -\frac{1}{4} \) is a parabola which is tangent to the line \( g = k_1 \) and \( h = k_2 \). Then grad \( f \) is perpendicular to this curve, and so are grad \( g \) and grad \( h \). With nine variables and six constraints, there will be six multipliers and eventually 15 equations. If a constraint is an inequality \( g \leq k \), then its multiplier must satisfy \( \lambda \leq 0 \) at a minimum.

\[ 1. f = x^2 + (k - 2x)^2; \quad \frac{df}{dx} = 2x - 4(k - 2x) = 0; \quad (\frac{2k}{3}, \frac{k}{3}) \quad \lambda = -4, x_{\text{min}} = 2, y_{\text{min}} = 2 \\
2. \frac{\lambda}{\partial x} = 0 \] and \( L = f_{\text{min}} \) leaves \( \frac{df}{dx} = \lambda \)

\[ 5. \lambda = \frac{1}{3}(1 + \sqrt{17}) \cdot (x, y) = (\pm\sqrt{2}, 0) \) or \( (0, \pm\sqrt{2}), f_{\text{min}} = 2^{1/3} \); \( \lambda = \frac{1}{3} \cdot (x, y) = (\pm 1, 1), f_{\text{max}} = 2 \\
7. \lambda = \frac{1}{2}, (x, y) = (2, -3); \text{tangent line} = 2x - 3y = 13 \\
9. (1 - c)^2 + (a - c)^2 + (2 - a - b - c)^2 + (2 - b - c)^2 \) is minimised at \( a = -\frac{1}{2}, b = \frac{3}{2}, c = \frac{3}{4} \)

11. \( (1, -1) \) and \( (-1, 1); \lambda = -\frac{1}{2} \)

13. \( f \) is not a minimum when \( C \) crosses to lower level curve; stationary point when \( C \) is tangent to level curve.

15. Substituting \( \frac{dL}{dx} = \frac{dL}{dy} = \frac{dL}{d\lambda} = 0 \) and \( L = f_{\text{min}} \) leaves \( \frac{df}{dx} = \lambda \)

\[ 17. x^2 \text{ is never negative; } (0, 0); 1 = \lambda(-3y^2) \text{ but } y = 0; g = 0 \text{ has a cusp at } (0, 0) \]

19. \( 2x = x_1 + x_2, 4y = 2x_1 + 2x_2, x + y + z = 0, x - z = 1 \) gives \( \lambda_1 = 0, \lambda_2 = 1, f_{\text{min}} = \frac{1}{2} \) at \( (2, 0, -\frac{1}{2}) \)

21. \( (1, 0, 0); (0, 1, 0); (\lambda_1, \lambda_2, 0); x = y = 0 \) \[ 25. (1, 0, 0), (0, 1, 0), (0, 0, 1); \text{at these points } f = 4 \text{ and } -2 \) (min) and \( 9 \) (max)

27. By increasing \( k \), more points are available so \( f_{\text{max}} \) goes up. Then \( \lambda = \frac{df}{dk} \geq 0 \)

29. \( (0, 0); \lambda = 0; f_{\text{min}} \text{ stays at } 0 \\
31. \lambda = 1 + \lambda_2, 6 = 1 + \lambda_3, \lambda_2 \geq 0, \lambda_3 \leq 0; \text{subtraction } 5 - 6 = \lambda_2 - \lambda_3 \text{ or } -1 \geq 0 \) (impossible); \[ x = 2004, y = -2000 \text{ gives } 5x + 6y = -1980 \]

33. \( 2x = 4\lambda_1 + \lambda_2, 2y = 4\lambda_1 + \lambda_3, \lambda_2 \geq 0, \lambda_3 \geq 0, 4x + 4y = 40; \text{max area } 100 \) at \( (10, 0)(0, 10); \text{min } 25 \) at \( (5, 5) \)

2. \( x^2 + y^2 = 1 \) and \( 2xy = \lambda(2x) \) and \( z^2 = \lambda(2y) \) yield \( 2\lambda^2 = \lambda^2 = 1 \). Then \( \lambda = \frac{1}{2} \) gives \( x_{\text{max}} = \pm \sqrt{3} \)

\( y_{\text{max}} = \sqrt{3} \), \( f_{\text{max}} = \frac{2\sqrt{3}}{3} \). Also \( \lambda = -\frac{1}{2} \) gives \( f_{\text{min}} = -\frac{2\sqrt{3}}{3} \)

4. \( x^2 + 9y^2 = 1 \) and \( 3 = \lambda(2x) \) and \( 1 = \lambda(18y) \) give \( \lambda = \lambda = \frac{1}{3} \) \( (x^2 + 9y^2) \) = 1 or \( \lambda^2 = \frac{1}{18} \). Then \( x_{\text{max}} = \frac{\sqrt{2}}{\sqrt{3}} \)

\( y_{\text{max}} = \frac{1}{3\sqrt{2}}, f_{\text{max}} = \frac{\sqrt{2} \sqrt{3}}{3} \). Change signs for \( (x, y, f_{\text{min}}) \). Second approach: Fix \( 3x + y \) and maximise \( x^2 + 9y^2 \).

6. \( 1 = \frac{3}{2} \left( \frac{x}{y} \right)^{1/3} \) and \( 1 = \frac{3}{2} \left( \frac{x}{y} \right)^{1/3} \) yield \( 1 = \frac{3}{2} \left( \frac{x}{y} \right)^{1/3} \) or \( \frac{1}{3} = \lambda(4)^{-1/3} \). Then \( \lambda = \left( \frac{3}{2} \right)^{1/3} = \frac{3}{2} \) so \( y = 2x \). The constraint gives \( x^1/3(2x) = k \) or \( x = k(4)^{-1/3} \) and then \( y = 2k(4)^{-1/3} \). Then \( f = x + y = 3k(4)^{-1/3} \).

8. \( \alpha = \lambda(2x), \beta = \lambda(2y), \gamma = \lambda(2z) \) give \( \frac{\partial f}{\partial k} = k^2 \) and \( \lambda = \sqrt{a^2 + b^2 + c^2} / 2k \). Then \( x_{\text{max}} = ak / \sqrt{a^2 + b^2 + c^2}, y_{\text{max}} = bk / \sqrt{a^2 + b^2 + c^2} \), and \( x_{\text{max}} = ck / \sqrt{a^2 + b^2 + c^2} \).

Thus \( \alpha, \beta, \gamma \cdot (x, y, z) \leq f_{\text{max}} = \sqrt{a^2 + b^2 + c^2} \) is the Schwarz inequality.

10. The base is \( b \), the triangle height is \( a \), the area is \( ab = \frac{1}{2}bh = 1 \).

Minimise \( f = b + 2a + 2\sqrt{b^2 + 4 + h^2} \). The \( \frac{\partial f}{\partial a}, \frac{\partial f}{\partial h} \) equations are \( 2 = \lambda b, \frac{2h}{\sqrt{b^2 + 4 + h^2}} = \lambda(\lambda b) \),

\[ 143 \]
13.7 Constraints and Lagrange Multipliers

The equation \( z^2 + y^2 = c^2 \) gives the circle of radius \( c \) in the plane. The multipliers \( \lambda \) and \( \mu \) point in the directions of unit vectors \( i_1 \) and \( i_2 \) tangent to the constraint surface. Moreover, \( \lambda_i = \mu_j = 0 \) for \( i \neq j \).

The Lagrange equations are:

\[
\begin{align*}
\frac{\partial f}{\partial x} - \lambda \frac{\partial g}{\partial x} = 0 \\
\frac{\partial f}{\partial y} - \mu \frac{\partial g}{\partial y} = 0
\end{align*}
\]

28 \( \lambda = 0 \) when \( h > k \) (not \( h = k \)) at the minimum. Reasoning: An increase in \( k \) leaves the same minimum.

Therefore \( f_{\min} \) is unchanged. Therefore \( \lambda = df_{\min}/dk \) is zero.

30 \( f = x^2 + y^2, x + y \geq 4 \) has minimum at \( x = y = 2 \). From \( 2x = \lambda(1) \) and \( 2y = \lambda(1) \), the multiplier is \( \lambda = 4 \) and \( f_{\min} = 8 \). Change to \( x + y \geq 4 + dk \). Then \( f_{\min} = 8 + \lambda dk = 8 + 4dk \). Check: \( x_{\min} = y_{\min} = \frac{1}{2}(4 + dk) \)

give \( f_{\min} = \left(\frac{1}{2}\right)^2(4 + dk)^2(2) = 8 + 4dk + \frac{1}{2}(dk)^2 \).

32 Lagrange equations: \( 2 = \lambda_1 + \lambda_2, 3 = \lambda_1 + \lambda_3, 4 = \lambda_1 + \lambda_4 \). Then \( \lambda_4 > \lambda_3 > \lambda_2 > 0 \). We need \( \lambda_4 > 0 \) and \( \lambda_3 > 0 \) (correction: not 0). Zero multiplier goes with nonzero \( x = 1 \). Nonzero multipliers go with \( y = z = 0 \). Then \( f_{\min} = 2 \). (We can see directly that \( f_{\min} = 2 \).)
Resource: Calculus Online Textbook
Gilbert Strang

The following may not correspond to a particular course on MIT OpenCourseWare, but has been provided by the author as an individual learning resource.

For information about citing these materials or our Terms of Use, visit: http://ocw.mit.edu/terms.