5.1 The Idea of the Integral (page 181)

The problem of summation is to add \( u_1 + \cdots + u_n \). It is solved if we find \( f \)'s such that \( u_j = f_j - f_{j-1} \). Then \( u_1 + \cdots + u_n \) equals \( f_n - f_0 \). The cancellation in \( (f_1 - f_0) + (f_2 - f_1) + \cdots + (f_n - f_{n-1}) \) leaves only \( f_n \) and \(-f_0\). Taking sums is the reverse (or inverse) of taking differences.

The differences between 0, 1, 4, 9 are \( u_1, u_2, u_3 \). For \( f_j = j^2 \) the difference between \( f_{10} \) and \( f_0 \) is \( u_{10} = 19 \). From this pattern \( 1 + 3 + 5 + \cdots + 19 \) equals 100.

For functions, finding the integral is the reverse of finding the derivative. If the derivative of \( f(x) \) is \( u(x) \), then the integral of \( u(x) \) is \( f(x) \). If \( u(x) = 10x \) then \( f(x) = 5x^2 \). This is the area of a triangle with base \( x \) and height 10x.

Integrals begin with sums. The triangle under \( v \) out to \( x = 4 \) has area 80. It is approximated by four rectangles of heights 10, 20, 30, 40 and area 100. It is better approximated by eight rectangles of heights 5, 10, \cdots, 40 and area 90. For \( n \) rectangles covering the triangle the area is the sum of \( \frac{40}{n} + \frac{80}{n} + \cdots + 40 \) = \( 80 + \frac{80}{n} \). As \( n \rightarrow \infty \) this sum should approach the number 80. That is the integral of \( v = 10x \) from 0 to 4.

1 \( 1, 3, 7, 15, 127 \) 3 \( -\frac{1}{2} - \frac{1}{4} - \frac{1}{8} = \frac{1}{8} \) 5 \( f_j - f_0 = \frac{t_j - 1}{t_j - 1} \) 7 \( 3x \) for \( x \leq 7, 7x - 4 \) for \( x \geq 1 \)
9 \( \frac{1}{52}, \frac{1}{2}, \frac{1}{2}, \sqrt{\frac{7}{2}} \) 11 Lower by 2 13 Up, down; rectangle 15 \( \sqrt{x + \Delta x} - \sqrt{x} \), \( \Delta x; \frac{f_j}{\delta x} \) \( \sqrt{x} \)
17 \( 6; 18; \) triangle 19 \( 18 \) rectangles 21 \( 6x - \frac{1}{2} x^2 - 10; 6 - x \) 23 \( \frac{14}{27} \) 25 \( x^2; x^2; \frac{1}{3} x^3 \)

2 (a) \( 2^5 - 2^4 = 16 = v_5 \) (b) \( 1 + 2 + 4 + 8 + 16 = f_5 - f_0 = 31 \)
4 Any \( C \) can be added to \( f(x) \) because the derivative of a constant is zero.
6 \( f_0 = \frac{1 + 1}{2 - 1} = 0; 1 + r + \cdots + r^n = f_n = \frac{r^{n+1} - 1}{r - 1} \).
8 The \( f \)'s are 0, 1, \(-1, 2, -2, \cdots \). Here \( u_j = (-1)^{j+1} j \) or \( v_j = \begin{cases} j & j \text{ odd} \\ -j & j \text{ even} \end{cases} \) and \( f_j = \begin{cases} \frac{j+1}{2} & j \text{ odd} \\ \frac{j}{2} & j \text{ even} \end{cases} \)
10 Within each quarter the sum over 13 weeks is lower than the single value for the whole quarter.
12 The last rectangle for the pessimist has height \( \sqrt{\frac{15}{4}} \). Since the optimist's last rectangle of area

\( \frac{1}{4} \sqrt{\frac{16}{4}} = \frac{1}{2} \)
is missed, the total area is reduced by \( \frac{1}{2} \).
14 The optimist's rectangles contain the curve. The pessimist's rectangles lie under the curve.
16 Under the \( \sqrt{x} \) curve, the first triangle has base 1, height 1, area \( \frac{1}{2} \). To its right is a rectangle of area 3.

Above the rectangle is a triangle of base 3, height 1, area \( \frac{3}{2} \). The total area \( \frac{1}{2} + 3 + \frac{3}{2} = 5 \) is below the curve.
18 The total rectangular area is 21.
20 The rectangles have area 2 times 5, 2 times 3, and 2 times 1, adding to 18. This is exactly correct because each overestimate is compensated by an equal underestimate.
22 The region is a right triangle with height \( 6 - x \) and base \( 6 - x \) and area \( \frac{1}{2}(6 - x)^2 \). This has derivative \( x - 6 \), which is \(-u(x)\) (minus sign because area decreases as \( x \) increases).
24 The areas under \( \sqrt{x} \) and under \( x^2 \) add to 1. The same is true for the areas under \( x^3 \) and \( x^{1/3} \).
5.2 Antiderivatives

Integration yields the area under a curve \( y = v(x) \). It starts from rectangles with the base \( \Delta x \) and heights \( v(x) \) and areas \( v(x) \Delta x \). As \( \Delta x \to 0 \) the area \( v_1 \Delta x + \cdots + v_n \Delta x \) becomes the integral of \( v(x) \). The symbol for the indefinite integral of \( v(x) \) is \( \int v(x) \, dx \).

The problem of integration is solved if we find \( f(x) \) such that \( \frac{df}{dx} = v(x) \). Then \( f \) is the antiderivative of \( v \), and \( \int_a^b v(x) \, dx \) equals \( f(b) \) minus \( f(a) \). The limits of integration are 2 and 6. This is a definite integral, which is a number and not a function \( f(x) \).

The example \( v(x) = x \) has \( f(x) = \frac{1}{2}x^2 \). It also has \( f(x) = \frac{1}{2}x^2 + 1 \). The area under \( v(x) \) from 2 to 6 is 16. The constant is canceled in computing the difference \( f(6) \) minus \( f(2) \). If \( v(x) = x^8 \) then \( f(x) = \frac{1}{9}x^9 \).

The sum \( v_1 + \cdots + v_n = f_n - f_0 \) leads to the Fundamental Theorem: \( \int_a^b v(x) \, dx = f(b) - f(a) \). The indefinite integral is \( f(x) \) and the definite integral is \( f(b) - f(a) \). Finding the area under the \( v \)-graph is the opposite of finding the slope of the \( f \)-graph.

**Exercise:** Area under \( B \) - area under \( D \); time when \( B = D \); time when \( B - D \) is largest

26 Choose \( v(x) \) to be positive until \( x = 1 \), zero to \( x = 2 \), then negative to \( x = 3 \). For total area 1,
5.3 Summation Versus Integration (page 194)

The Greek letter $\sum$ indicates summation. In $\sum_{j=1}^{n} v_j$ the dummy variable is $j$. The limits are $j = 1$ and $j = n$, so the first term is $v_1$ and the last term is $v_n$. When $v_j = j$ this sum equals $\frac{1}{2}n(n + 1)$. For $n = 100$ the leading term is $\frac{1}{2}100^2 = 5000$. The correction term is $\frac{1}{2}n = 50$. The leading term equals the integral of $v = x$ from 0 to 100, which is written $\int_{0}^{100} x \, dx$. The sum is the total area of 100 rectangles. The correction term is the area between the sloping line and the rectangles.

The sum $\sum_{i=3}^{5} i^2$ is the same as $\sum_{j=1}^{4} (j + 2)^2$ and equals 86. The sum $\sum_{i=4}^{5} v_i$ is the same as $\sum_{i=0}^{1} u_{i+4}$ and equals $v_4 + v_5$. For $f_n = \sum_{j=1}^{n} u_j$ the difference $f_n - f_{n-1}$ equals $v_n$.

The formula for $1^2 + 2^2 + \cdots + n^2$ is $f_n = \frac{1}{6}n(n + 1)(2n + 1) = \frac{1}{3}n^3 + \frac{1}{2}n^2 + \frac{1}{6}n$. To prove it by mathematical induction, check $f_1 = 1$ and check $f_n - f_{n-1} = n^2$. The area under the parabola $v = x^2$ from $x = 0$ to $x = 9$ is $\frac{1}{8}9^3$. This is close to the area of $9/\Delta x$ rectangles of base $\Delta x$. The correction terms approach zero very slowly.

1 $\frac{45}{12} ; 16$ 2 $2n^2 + 1$ 3 $\sum_{i=0}^{n} a_i x^i$ 4 $\sum_{i=1}^{n} \sin \frac{2\pi j}{n}$ 5 $\sum_{i=1}^{50} 2j = 2550$ 6 $\sum_{i=1}^{50} (2j - 1) = 2500$ 7 $\sum_{i=1}^{4} (-1)^{k+1}/k = \frac{7}{12}$

8 $\int_{-\frac{1}{2}}^{a+b} \int_{-\frac{b}{2}}^{a+b} C; f_0 - f_0 - f_1 + f_0$ 9 $f_1 = 1; n^2 + (2n + 1) = (n + 1)^2$ 10 $a + b + c = 5, 3a + 9b + 27c = 14; \text{sum of squares}$ 11 $2(a^2 + b^2) = 13$ 12 $2n^2 + 1; \frac{1}{11} - \frac{1}{11}$ 13 $\sum_{i=1}^{15} f_i$; 14 $\frac{d^2 f}{dx^2}$ 15 $\sum_{j=1}^{7} v(j \Delta z)$ 16 $\Delta z \sum_{j=1}^{n} v(j \Delta z)$ 17 $f(1) - f(0) = f_0 \frac{d^2 f}{dx^2} \Delta x$ 18 $\sum_{j=1}^{6} (-1)^{j+1} v_j$; 19 $\sum_{i=1}^{n} v_i w_i$; $\sum_{i=1}^{3} v_{i-1}$ 20 $\sum_{i=1}^{n} v_{i-1}$ 21 $\sum_{j=1}^{n} \sin \frac{2\pi j}{n}$ 22 $\sum_{j=1}^{n} a^n - b^j$ 23 $S_{400} = 80200; E_{400} = .0025 = \frac{1}{n}$ 24 $S_{100,1/3} \approx 350, E_{100,1/3} \approx .00587; S_{100,3} = 25502500, E_{100,3} = .0201$ 25 $v_1 = 9, v_2 = 12, \Sigma \Sigma = 21$ 26 At $N = 1, 2^{N-2}$ is not 1 27 $0; \frac{1}{n} (v_1 + \cdots + v_n)$ 28 $\Delta z \sum_{j=1}^{n} v(j \Delta z)$ 29 $f(1) - f(0) = f_0 \frac{d^2 f}{dx^2} \Delta x$ 30 The sums approach $\approx .36788$; the sum is close to $e^{-1} = 2.71828$; the product is extremely near 1. 31 Choose all $a$'s and $b$'s equal to 1. Then $n^2 \neq n$. 32 $f_n - f_0$ and $f_{12} - f_3$ (by telescoping: the other terms cancel). 33 $\sum_{i=1}^{6} v_i = \sum_{i=1}^{10} v_{i+1}$ and $\sum_{i=0}^{n} i^2 = \sum_{i=2}^{n} (1 - 2)^2$. 34 The sums are $-1, 1, -2, 2, \cdots$ and the sum up to $n = 6$ is 3.

The first sum is close to $e^{-1} = .36788$; the second is close to $e = 2.71828$; the product is extremely near 1. 35 Choose all $a$'s and $b$'s equal to 1. Then $n^2 \neq n$. 36 $f_n - f_0$ and $f_{12} - f_3$ (by telescoping: the other terms cancel). 37 $\sum_{i=1}^{6} v_i = \sum_{i=1}^{10} v_{i+1}$ and $\sum_{i=0}^{n} i^2 = \sum_{i=2}^{n} (1 - 2)^2$. 38 $\sum_{j=1}^{n} v_j$; 39 $\sum_{i=1}^{n} v_i w_i$; 40 $\sum_{i=1}^{n} v_{i-1}$ 41 $\sum_{j=1}^{n} \sin \frac{2\pi j}{n}$ 42 $\sum_{j=1}^{n} a^n - b^j$ 43 44 The sums approach $\approx .36788$; the sum is close to $e^{-1} = 2.71828$; the product is extremely near 1. 45 Choose all $a$'s and $b$'s equal to 1. Then $n^2 \neq n$. 46 $f_n - f_0$ and $f_{12} - f_3$ (by telescoping: the other terms cancel). 47 $\sum_{i=1}^{6} v_i = \sum_{i=1}^{10} v_{i+1}$ and $\sum_{i=0}^{n} i^2 = \sum_{i=2}^{n} (1 - 2)^2$. 48 49 The sums approach $\approx .36788$; the sum is close to $e^{-1} = 2.71828$; the product is extremely near 1. 50 Choose all $a$'s and $b$'s equal to 1. Then $n^2 \neq n$. 51 $f_n - f_0$ and $f_{12} - f_3$ (by telescoping: the other terms cancel). 52 $\sum_{i=1}^{6} v_i = \sum_{i=1}^{10} v_{i+1}$ and $\sum_{i=0}^{n} i^2 = \sum_{i=2}^{n} (1 - 2)^2$. 53
5.4 Indefinite Integrals and Substitutions

Finding integrals by substitution is the reverse of the chain rule. The derivative of \((\sin x)^3\) is \(3(\sin x)^2 \cos x\). Therefore the antiderivative of \(3(\sin x)^2 \cos x\) is \((\sin x)^3\). To compute \(\int (1 + \sin x)^2 \cos x \, dx\), substitute \(u = 1 + \sin x\). Then \(du/dx = \cos x\) so substitute \(du = \cos x \, dx\). In terms of \(u\) the integral is \(\int u^2 \, du = \frac{1}{3} u^3\). Returning to \(x\) gives the final answer.

The best substitutions for \(\int \tan(x+3) \sec^2(x+3) \, dx\) and \(\int (x^2 + 1)^{10} x \, dx\) are \(u = \tan(x+3)\) and \(u = x^2 + 1\). Then \(du = \sec^2(x+3) \, dx\) and \(2x \, dx\). The answers are \(\frac{1}{2} \tan^2(x+3)\) and \(\frac{1}{22}(x^2 + 1)^{11}\). The antiderivative of \(v \, du/dx\) is \(\frac{1}{2} v^2\). \(\int 2x \, dx/(1 + x^2)\) leads to \(\int \frac{du}{u}\), which we don’t yet know. The integral \(\int dx/(1 + x^2)\) is known immediately as \(\tan^{-1}x\).

\[
\begin{align*}
12 \quad & \frac{2}{3}(2 + x)^{3/2} + C \\
3 \quad & (x + 1)^{n+1}/(n + 1) + C(n \neq -1) \\
9 \quad & -\frac{1}{8} \cos^4 2x + C \\
11 \quad & \sin^{-1} t + C \\
13 \quad & \frac{1}{2}(1 + t^2)^{3/2} - (1 + t^3)^{1/2} + C \\
15 \quad & 2\sqrt{x} + x + C \\
17 \quad & \sec x + x + C \\
21 \quad & \frac{1}{2}x^2 + \frac{3}{2}x^{3/2} \\
23 \quad & -\frac{1}{2}(1 - 2x)^{3/2} \\
25 \quad & y = \sqrt{2x} \\
27 \quad & \frac{1}{2}x^2 \\
29 \quad & a \sin x + b \cos x \\
31 \quad & \frac{1}{16}x^{5/2} \\
33 \quad & F; F; F; F \\
35 \quad & f(x - 1); 2f(\frac{3}{2}) \\
37 \quad & x - \tan^{-1} x \\
39 \quad & \frac{1}{u} \, du \\
41 \quad & 4.9t^2 + C \\
43 \quad & f(t + 3); f(t); 3f(t); \frac{1}{3} f(3t)
\end{align*}
\]

\[
\begin{align*}
2 \quad & \frac{-2}{3}(3 - x)^{3/2} + C \\
4 \quad & \frac{1}{1-n}(x + 1)^{1-n} - n, for n \neq 1. \\
6 \quad & \frac{2}{9}(1 - 3x)^{3/2} + C \\
8 \quad & \frac{1}{2} \sin^2 x + C or -\frac{1}{2}(\sin x)^2 - C \\
10 \quad & \cos^3 x \sin 2x equals 2 \cos^4 x \sin x and its integral is \frac{-3}{2} \cos^3 x + C \\
12 \quad & \frac{1}{3}(1 - t^2)^{3/2} + C \\
14 \quad & Write u = 1 - x^2 and \(du = -2x \, dt\) to give \(\int (1-u)\sqrt{u} \, du = -\frac{1}{3}u^{3/2} + \frac{1}{2}u^{5/2} + C = -\frac{1}{3}(1 - t^2)^{3/2} + \frac{1}{2}(1 - t^2)^{5/2} + C \\
16 \quad & The integral of \(x^{1/2} + x^2\) is \(\frac{2}{3}x^{3/2} + \frac{1}{6}x^3 + C. \\
18 \quad & Set u = \tan x and \(du = \sec^2 x \, dx\). The integral of \(u^2 \, du\) is \(\frac{1}{3} \tan^3 x + C. \\
20 \quad & Write \sin^3 x as \((1 - \cos x) \sin x\) and \(\cos^3 x \sin x\) and \(\sin x \sin x\) give \(\frac{1}{2} \cos^3 x - \cos x + C. \\
22 \quad & Substitute y = cx^n to find \text{nez}^{n-1} = (cx^n)^2. Match exponents: n - 1 = 2n or n = -1. Match coefficients: nc = c^2 or c = n = -1. Answer y = -1/x. \\
24 \quad & y = -\sqrt{1 - 2x} + C \\
26 \quad & dy/dx = y \text{ gives } dy = x \, dx or y^2 = x^2 + C or y = \sqrt{x^2 + C.}
\end{align*}
\]
28 \( y = \frac{1}{120}z^5 + C_1z^4 + C_2z^3 + C_3z^2 + C_4z + C_5 \)

30 \( y = \frac{3}{5}x^3 \) comes from \( y^{-1/2}dy = x^{1/2}dx \) or \( 2y^{1/2} = \frac{3}{2}x^{3/2}(+C) \)

32 \( \frac{dy}{dx} = x^{1/4} \) gives \( y = \frac{x^{5/4}}{5} + C \)

34 (a) False: The derivative of \( \frac{1}{2}f^2(x) \) is \( f(x) \frac{df}{dx} \) 
(b) True: The chain rule gives \( \frac{d}{dx}f(u(x)) = \frac{df}{du} \frac{du}{dx} \)

times \( \frac{d}{dx} \) (c) False: These are operations not inverse functions and (d) is True.

36 \( \frac{1}{2}(2x - 1) + C; \frac{1}{2}(2x^2 + x) + C \)

38 \( f(x^2 + 2x + 1)dx = \frac{1}{3}x^3 + \frac{2}{3}x^2 + x + C. \)

40 Use \( u = 1 + x^2 \) and \( du = 2x \, dx \) and \( x^2 = u - 1. \) Then \( \int \frac{du}{u} - \int \frac{du}{u} \) is \( \frac{1}{u} + \frac{1}{2u} + C = \frac{1}{1+u} + \frac{1}{2(1+u^2)^3} + C. \)

42 \( y = C_1x^3 + C_2x^2 + C_3x + C_4. \)

5.5 The Definite Integral (page 205)

If \( \int_a^b v(x) \, dx = f(x) + C, \) the constant \( C \) equals \( -f(a). \) Then at \( x = a \) the integral is zero. At \( x = b \) the integral becomes \( f(b) - f(a). \) The notation \( f(z) \) means \( f(b) - f(a). \) Also \( |x + 3| \) equals \( -2, \) which shows why the antiderivative includes an arbitrary constant. Substituting \( u = 2z - 1 \) changes \( \int_1^5 \sqrt{2z - 1} \, dz \) into \( \int_1^5 \frac{1}{2} \sqrt{u} \, du \) (with limits on \( u \)).

The integral \( \int_a^b v(x) \, dx \) can be defined for any continuous function \( v(x), \) even if we can't find a simple antiderivative. First the meshpoints \( x_1, x_2, \ldots \) divide \( [a, b] \) into subintervals of length \( \Delta x_k = x_k - x_{k-1}. \) The upper rectangle with base \( \Delta x_k \) has height \( M_k = \max v(x) \) in interval \( k. \)

The upper sum \( S \) is equal to \( \Delta x_1 M_1 + \Delta x_2 M_2 + \cdots. \) The lower sum \( s \) is \( \Delta x_1 m_1 + \Delta x_2 m_2 + \cdots. \) The area is between \( s \) and \( S. \) As more meshpoints are added, \( S \) decreases and \( s \) increases. If \( S \) and \( s \) approach the same limit, that defines the integral. The intermediate sums \( S*, \) named after Riemann, use rectangles of height \( v(x^*_k). \) Here \( x^*_k \) is any point between \( x_{k-1} \) and \( x_k, \) and \( S* = \sum \Delta x_k v(x^*_k) \) approaches the area.

1 \( C = f(2) \)

3 \( C = f(3) \)

5 \( f(t) \) is wrong

7 \( C = 0 \)

9 \( C = 0 \)

11 \( u = x^2 + 1; f_1^2 u^{10} \, du = \frac{t^{11} - 1}{22} = \frac{t^{11} - 1}{22} \)

13 \( u = \tan z; \int_0^1 \sin u \, du = \frac{1}{2} \)

15 \( u = \sec z; \int_1^2 u \, du = \frac{1}{2} \) (same as 13)

17 \( u = \frac{1}{z}, x = \frac{1}{u}, dx = -\frac{1}{u^2}, \int_1^2 \frac{du}{u^2} = \frac{1}{2} \)

19 \( S = \frac{1}{2}(\frac{1}{4} + 1)^4 + \frac{1}{4}(1 + 1)^4; s = \frac{1}{2}(0) + \frac{1}{4}(1 + 1)^4 \)

21 \( S = \frac{1}{2}(\frac{1}{4})^3 + (\frac{1}{4})^3 + (\frac{3}{4})^3 + (\frac{3}{4})^3 \)

23 \( S = \frac{1}{4}(\frac{1}{16})^4 + (\frac{1}{4})^4 + (\frac{3}{4})^4 + (\frac{3}{4})^4 + (\frac{3}{4})^4 \)

25 \( \frac{1}{2}(1 + 1 + \frac{1}{4} + \frac{1}{16} + \cdots) = \frac{1}{2}(\frac{1}{4} + \frac{1}{16} + \cdots) \)
12 Choose \( u = \sin x \). Then \( u = 0 \) at \( x = 0 \) and \( u = 1 \) at \( x = \frac{\pi}{2} \). The integral is \( \int_0^1 u^0 \, du = \left[ \frac{1}{2} u^0 \right]_0^1 = \frac{1}{2} \).

14 \( u = x^2 \) has \( du = 2x \, dx \); \( u = 0 \) at \( x = 0 \) and \( u = 4 \) at \( x = 2 \); then \( \int_0^2 x^2 \, x \, dx = \int_0^4 u^2 \, du = \frac{4^{n+1}}{2^{(n+1)}} \).

16 Choose \( u = x^2 \) with \( du = 2x \, dx \) and \( u = 0 \) at \( x = 0 \) and \( u = 1 \) at \( x = 1 \). Then \( \int_0^1 \frac{du}{2\sqrt{1-u}} = -\sqrt{1-u} \bigg|_0^1 = +1 \).

(Could also choose \( u = 1 - x^2 \).

18 With \( u = 1 - x \) and \( du = -dx \) the limits are \( u = 1 \) at \( x = 0 \) and \( u = 0 \) at \( x = 1 \). The integral \( \int_0^1 v(z) \, dx \) becomes \( \int_0^1 (1 - u)^3 u^3 (-du) \). Reverse limits by Property 3 on the next page: \( \int_0^1 (1 - u)^3 u^3 \, du \) which is the same as the original. 

20 \( \sin 2\pi x \) has maximum \( M_1 = 1 \) and minimum \( m_1 = 0 \) in the interval to \( x = \frac{1}{2} \); then \( M_2 = 0 \) and \( m_2 = -1 \) in the interval to \( x = 1 \). Thus \( S = \frac{1}{2} (1) \) and \( s = \frac{1}{2} (-1) \).

22 Maximum of \( x \) in the four intervals is: \( M_k = -\frac{1}{2}, 0, \frac{1}{2} \). Minimum is \( m_k = -1, -\frac{1}{2}, 0 \). Then \( S = \frac{1}{2} (-\frac{1}{2} + 0 + \frac{1}{2} + 1) = \frac{1}{2} \) and \( s = \frac{1}{2} (-1 - \frac{1}{2} + 0 + \frac{1}{2}) = -\frac{1}{2} \).

24 The exact area is \( \int_0^1 x^2 \, dx = \frac{4}{3} \). Then \( S - s = 2^3 \Delta x \). So \( S < 4.001 \) if \( 2^3 \Delta x < .001 \) or \( \Delta x < \frac{1}{8} (.001) = .000125 \).

26 All midpoints of the intervals with \( \Delta x = \frac{1}{n} \) are fractions. So \( V(x^*) = 1 \) at these midpoints \( x^* \).

The upper Riemann sum \( S^* \) is the sum of \( \Delta x \)'s times \( 1 \) length of interval of integration. This stays the same as \( n \to \infty \) but other choices of \( x^* \) give \( S^* = 0 \): not Riemann integrable.

28 (Correction: Change \( v \) to \( M \).) The graph of \( M(z) \) is above horizontal rectangles of total area 
\[ \left( \frac{1}{2} \right) (\frac{1}{2}) + \left( \frac{1}{4} \right) (\frac{1}{4}) + \cdots = \frac{1}{2^2} = \frac{1}{2} \]. With \( \Delta z = 3 \) the \( M \)'s are 0, 1, 1 with \( S = \frac{1}{2} (0 + 1 + 1) = \frac{1}{2} \).

The \( m \)'s are 0, \( \frac{1}{2} \) with \( s = \frac{1}{3} (0 + 0 + \frac{1}{2}) = \frac{1}{6} \).

30 Check \( f(1) = \int_0^1 v(x) \, dx = 0 \). Check \( \frac{d}{dx} \int_0^x v(x) \, dx = \frac{d}{dx} (\int_0^x v(x) \, dx) = -v(x). \) Then \( f(x) \) is correct.

5.6 Properties of the Integral and Average Value (page 212)

The integrals \( \int_0^b v(x) \, dx \) and \( \int_b^0 v(x) \, dx \) add to \( \int_0^b v(x) \, dx \). The integral \( \int_0^1 v(x) \, dx \) equals \( -\int_1^0 v(x) \, dx \). The reason is that the steps \( \Delta x \) are negative. If \( v(x) \leq x \) then \( \int_0^1 v(x) \, dx \leq \frac{1}{2} \). The average value of \( v(x) \) on the interval \( 1 \leq x \leq 9 \) is defined by \( \frac{1}{8} \int_0^9 v(x) \, dx \). It is equal to \( v(c) \) at a point \( x = c \) which is between \( 1 \) and \( 9 \).

The rectangle across the interval with height \( v(c) \) has the same area as the region under \( v(x) \). The average value of \( v(x) = x + 1 \) on the interval \( 1 \leq x \leq 9 \) is 6.

If \( x \) is chosen from 1, 3, 5, 7 with equal probabilities \( \frac{1}{4} \), its expected value (average) is 4. The expected value of \( x^2 \) is 21. If \( x \) is chosen from 1, 2, 3, 4 with probabilities \( \frac{1}{8} \), its expected value is 4.5. If \( x \) is chosen from 1 ≤ \( x \) ≤ 9, the chance of hitting an integer is zero. The chance of falling between \( x \) and \( x + dx \) is \( p(x) \, dx = \frac{1}{8} \, dx \). The expected value \( E(x) \) is the integral \( \int_1^9 \frac{2}{8} \, dx \). It equals 5.

1 \( v = \frac{1}{2} \left( \int_{-1}^1 x^4 \, dx \right) = \frac{1}{6} \) equals c^4 at \( c = \pm (\frac{1}{8})^{1/4} \)

3 \( v = \frac{1}{6} \int_0^\pi \cos^2 x \, dx = \frac{1}{2} \) equals \( \frac{2}{9} \) at \( c = \frac{

5 \( v = \int_1^2 x^5 \, dx = \frac{1}{6} \) equals 1 \( \sqrt{2} \) at \( c = \sqrt{2} \)

7 \( f_3^0 v(x) \, dx \)

9 False, take \( v(x) < 0 \)

11 True; \( \int_0^1 v(x) \, dx + \frac{3}{2} \cdot \int_0^2 v(x) \, dx = \frac{1}{2} \int_0^3 v(x) \, dx \)

13 False; when \( v(x) = x^2 \) the function \( x^2 - \frac{1}{3} \) is even

15 False; take \( v(x) = 1 \); factor \( \frac{1}{2} \) is missing

17 \( v = \frac{1}{6-a} \int_a^b v(x) \, dx \)

19 0 and \( \frac{2}{\pi} \)
21 \( v(x) = Cx^2; u(x) = C \). This is "constant elasticity" in economics (Section 2.2)

23 \( \bar{V} \to 0; \bar{Q} \to 1 \)

25 \( \frac{1}{2} \int_0^a (a-x) \, dx = a + 1 \) if \( a > 2; \frac{1}{2} \int_0^a |a-x| \, dx = \frac{a}{2} \) area = \( \frac{a^2}{2} - a + 1 \) if \( a < 2 \); distance = absolute value

27 Small interval where \( y = \sin \theta \) has probability \( \frac{d\theta}{\pi} \); the average \( y \) is \( \int_0^x \sin \theta \, d\theta = \frac{2}{\pi} \).

29 Area under \( \cos \theta \) is 1. Rectangle \( 0 \leq \theta \leq \frac{\pi}{2}, 0 \leq y \leq 1 \) has area \( \frac{\pi}{2} \). Chance of falling across a crack is \( \frac{1}{r/2} = \frac{2}{\pi} \).

31 \( \frac{1}{6}, \frac{1}{3}, \ldots, \frac{1}{3}; 10.5 \)

33 \( \frac{1}{4} \int_0^x 2x \cos \frac{2x}{2} \, dt = \frac{1}{4} \cdot 220 \cdot \frac{60}{2\pi} = V_{\text{ave}} \)

35 Any \( v(x) = v_{\text{even}}(x) + v_{\text{odd}}(x); (x + 1)^2 = (3x^2 + 1) + (x^2 + 3x) = \frac{1}{x+1} = 1-\frac{1}{x+1} - \frac{1}{x+3} \)

41 \( f(x) = \left\{ \begin{array}{ll} \frac{1}{2}(x-2)^2 & \text{if } x \geq 2 \\ -\frac{1}{2}(x-2)^2 & \text{if } x \leq 2 \end{array} \right. + C; f(5) - f(0) = \frac{9}{2} + \frac{1}{2} = \frac{13}{2} \)

2 \( v_{\text{ave}} = \frac{1}{5} \int_0^x x^5 \, dx = 0 \) which equals \( c^5 \) at \( c = 0 \).

4 \( v_{\text{ave}} = \frac{1}{4} \int_0^x \sqrt{x} \, dx = \frac{1}{4} \frac{5}{2} \) which equals \( \sqrt{c} \) at \( c = \frac{1}{4} \frac{5}{2} \).

6 \( v_{\text{ave}} = \frac{1}{2} \int_0^x \sin x \, dx = 0 \) (odd function over symmetric interval \(-\pi \) to \( \pi \)). This equals \( (\sin c)^9 \) at \( c = -\pi \) and \( 0 \) and \( \pi \).

8 \( \frac{1}{3} \int_0^x x^3 \, dx = x^3 \).

10 False. The interval keeps length 3 but if \( u(x) = x \) the integral changes.

12 False: This is the average value of \( f(x) \).

14 False: \(-1 \leq \sin x \leq 1 \) but the derivatives do not satisfy \( 0 \leq \cos x \leq 0 \).

16 (a) False: strictly speaking the antiderivatives of \( x^2 \) are \( ix^3 \) + \( C \); this is odd only when \( C = 0 \)

(b) False: \( x^3 \) is even.

18 The average of \( \frac{d^2}{dx^2} \) is \( \frac{1}{6} \cdot \frac{1}{2} = -1 \).

20 Property 6 proves both (a) and (b) because \( u(x) = |u(x)| \) and also \( -u(x) \leq |u(x)| \). So their integrals maintain these inequalities.

22 If \( u \) is increasing then \( u(t) \leq u(x) \) when \( t \leq x \). Apply Property 6: \( \int_a^x u(t) \, dt \leq \int_a^x u(t) \, dt \). Note \( v(x) \) is constant in the last integral, which is \( t v(x) |x| = 2u(x) \).

24 Suppose \( v_n < s \) for \( n \) larger than \( N \). \( (N \) is now fixed. \) Then the average \( \frac{v_1 + \ldots + v_n}{n} \) is less than \( \frac{v_1 + \ldots + v_n + (n-N)s}{n} \).

As \( n \to \infty \) this approaches \( \frac{v_n}{N} = \varepsilon \). So the average goes below any \( \varepsilon \) and must approach zero.

28 \( \bar{y}_{\text{ave}} = \frac{1}{3} \int_0^x (x-2)^2 \, dx = \frac{1}{3} \) (area = \( \frac{1}{3} \)). A uniform distribution of \( Q \) along the base is different from a uniform distribution of \( P \) along the semicircle.

30 This needle falls across a crack when \( y < x \cos \theta \) (change the 1's in the Buffon needle figure to \( x \)'s).

Following Problem 29, the shaded region lies under \( y = x \cos \theta \) and under \( y = 1 \).

Keeping \( x < 1 \) (shorter needles only) the area is \( \int_0^x \cos \theta \, d\theta = x \sin \theta \). The fraction \( \frac{\pi}{x} = \frac{2\pi}{x} \) of the total area is the probability of falling across a crack.

32 The square has area 1. The area under \( y = \sqrt{x} \) is \( \int_0^1 \sqrt{x} \, dx = \frac{2}{3} \).

34 When \( x \) is replaced by \(-x \), the function \( \frac{1}{4} \left( u(x) + u(-x) \right) \) is unchanged (even). The function \( \frac{1}{4} \left( u(x) - u(-x) \right) \) becomes \( \frac{1}{4} \left( u(-x) - u(x) \right) \) so signs are reversed (odd function).

36 \( f'(-x) = \lim_{{h \to 0}} \frac{f(x+h)-f(-x)}{h} \) (when \( f \) is even) \( \lim_{{h \to 0}} \frac{f(x-h)-f(-x)}{h} = -f'(x) \). Thus \( f' \) is odd.

38 Average size is \( \frac{C}{n} \). The chance of an individual belonging to group 1 is \( \frac{x_1}{n} \). The expected size is sum

of size times probability: \( E(z) = \sum_1^n \frac{x_1}{n} \). This exceeds \( \frac{C}{n} \) by the Schwarz inequality:

\( (1x_1^2 + \ldots + 1x_n^2)^2 \leq (1^2 + \ldots + 1^2)(x_1^2 + \ldots + x_n^2) \) is the same as \( C^2 \leq n \sum x_i^2 \).

40 This formula for \( f(x) \) jumps from 9 to -9. The correct formula (with continuous \( f \)) is \( x^2 \) then \( 18 - x^2 \).

Then \( f(4) - f(0) = 2 \), which is \( \int_0^x u(x) \, dx \).

42 The integral of \( u(x) - u_{\text{ave}} \) is zero (equal positive and negative areas): \( \int_a^b u_{\text{ave}} \, dx = (b-a)v_{\text{ave}} = \int_a^b u(x) \, dx \).
5.7 The Fundamental Theorem and Its Applications (page 219)

The area \( A(z) = \int_a^z v(t) \, dt \) is a function of \( x \). By Part 1 of the Fundamental Theorem, its derivative is \( v(x) \). In the proof, a small change \( \Delta x \) produces the area of a thin rectangle. This area \( \Delta f \) is approximately \( \Delta x \) times \( v(x) \). So the derivative of \( \int_a^z t^2 \, dt \) is \( x^2 \).

The integral \( \int_a^b t^2 \, dt \) has derivative \(-x^2\). The minus sign is because \( x \) is the lower limit. When both limits \( a(z) \) and \( b(z) \) depend on \( z \), the formula for \( \frac{d}{dz} f \) becomes \( v(b(x)) \frac{db}{dx} - v(a(x)) \frac{da}{dx} \). In the example \( \int_0^2 t^2 \, dt \), the derivative is \( 2x \).

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5.8 Numerical Integration (page 226)

To integrate \( y(x) \), divide \([a, b]\) into \( n \) pieces of length \( \Delta x = (b - a)/n \). \( R_n \) and \( L_n \) place a rectangle over each piece, using the height at the right or left endpoint:

\[
R_n = \Delta x(y_1 + \cdots + y_n) \quad \text{and} \quad L_n = \Delta x(y_0 + \cdots + y_{n-1}).
\]

These are first-order methods, because they are incorrect for \( y = x \). The total error on \([0, 1]\) is approximately 

\[
\frac{1}{2} \epsilon(1) - y(0)). \quad \text{For} \quad y = \cos 2\pi x \quad \text{the error is very small because} \quad [0, 1] \quad \text{is a complete period.}
\]

A much better method is \( T_n = \frac{1}{2} R_n + \frac{1}{2} L_n = \Delta x[y_0 + 2y_1 + \cdots + y_{n-1}] \). This trapezoidal rule is second-order because the error for \( y = x \) is zero. The error for \( y = x^2 \) from \( a \) to \( b \) is 

\[
\frac{1}{6} (\Delta x)^2 (b - a).
\]

The midpoint rule is twice as accurate, using 

\[
M_n = \Delta x[y_0 + \cdots + y_{n-1}].
\]

Simpson’s method is 

\[
S_n = \frac{2}{3} M_n + \frac{1}{3} T_n. \quad \text{It is fourth-order, because the powers} \quad 1, x, x^2, x^3 \quad \text{are integrated correctly. The coefficients of} \quad y_0, y_1, y_2, y_3 \quad \text{are} \quad \frac{1}{6}, \frac{4}{6}, \frac{2}{6}, \frac{1}{6} \quad \text{times} \quad \Delta x. \quad \text{Over} \quad 3 \quad \text{intervals the weights are} \quad \Delta x/6 \quad \text{times} \quad 1 - 4 - 2 - 4 - 2 - 4 - 1. \quad \text{Gauss uses} \quad \text{two} \quad \text{points in each interval, separated by} \quad \Delta x/\sqrt{3}. \quad \text{For a method of order} \quad p \quad \text{the error is nearly proportional to} \quad (\Delta x)^p.
\]

which agrees with \( A = s^2 \).

32 The 4-dimensional cube has volume \( H = s^4 \). The face with \( x = s \) is a 3-dimensional cube. Its volume is 

\[
V = s^3. \quad \text{Four faces have volume} \quad 4s^3. \quad \text{Increase by} \quad \Delta s \quad \text{gives} \quad \Delta H = (s + \Delta s)^4 - s^4. \quad \text{So} \quad dH/ds = 4s^3.
\]

34 \( \int_z dy = \int_0^1 \sqrt{y} \, dy = \frac{2}{3} y^{3/2} \big|_0^1 = \frac{2}{3} \).

36 \( A \) is the area under \( y = \sqrt{r^2 - x^2} \) (quarter of a circle). Then 

\[
\int_0^{\pi/2} \sqrt{r^2 - x^2} \, dx = \int_0^{\pi/2} r \cos \theta \, (r \cos \theta \, d\theta) = \frac{\pi}{4} r^2 \quad \text{because the average value of} \quad \cos^2 \theta \quad \text{is} \quad \frac{1}{2}. \quad \text{Its integral is} \quad \frac{\pi}{4} (\theta + \sin \theta \cos \theta) \big|_0^{\pi/2} = \frac{\pi}{4}.
\]

38 The triangle ends at the line \( x + y = 1 \) or \( r \cos \theta + r \sin \theta = 1 \). The area is \( \frac{\pi}{2} \), by geometry. So the area 

\[
\int_0^\pi \int_0^1 \sqrt{r^2 - x^2} \, d\theta \, dr = \frac{\pi}{2}.
\]

34 \( \int_0^1 y \, dy = \int_0^1 \sqrt{y} \, dy = \frac{2}{3} y^{3/2} \big|_0^1 = \frac{2}{3} \).

36 \( A \) is the area under \( y = \sqrt{r^2 - x^2} \) (quarter of a circle). Then 

\[
\int_0^{\pi/2} \sqrt{r^2 - x^2} \, dx = \int_0^{\pi/2} r \cos \theta \, (r \cos \theta \, d\theta) = \frac{\pi}{4} r^2 \quad \text{because the average value of} \quad \cos^2 \theta \quad \text{is} \quad \frac{1}{2}. \quad \text{Its integral is} \quad \frac{\pi}{4} (\theta + \sin \theta \cos \theta) \big|_0^{\pi/2} = \frac{\pi}{4}.
\]

38 The triangle ends at the line \( x + y = 1 \) or \( r \cos \theta + r \sin \theta = 1 \). The area is \( \frac{\pi}{2} \), by geometry. So the area 

\[
\int_0^\pi \int_0^1 \sqrt{r^2 - x^2} \, d\theta \, dr = \frac{\pi}{2}.
\]

40 Rings have area \( 2\pi r \, dr \), and 

\[
\int_0^1 2\pi r \, dr = \pi r^2 \big|_0^1 = \pi.
\]

42 The strip around the ellipse does not have constant width \( dr \). The width is \( dr \) in the \( x \) direction and \( 2 \, dr \) in the \( y \) direction.

44 The sum to \( j = n \) of the differences \( f_j - f_{j-1} \) is \( f_n + C \) (and the constant is \( C = -f_0 \)). This sum telescopes: 

\[
(f_1 - f_0) + (f_2 - f_1) + (f_3 - f_2) \cdots
\]

46 The strip around the ellipse does not have constant width \( dr \). The width is \( dr \) in the \( x \) direction and \( 2 \, dr \) in the \( y \) direction.

44 The sum to \( j = n \) of the differences \( f_j - f_{j-1} \) is \( f_n + C \) (and the constant is \( C = -f_0 \)). This sum telescopes: 

\[
(f_1 - f_0) + (f_2 - f_1) + (f_3 - f_2) \cdots
\]

48 At \( t = 1 \) the area is under the parabola \( y = -x^2 + 1 \). The line along the base has length \( \sqrt{2} \), because an increase \( At \) raises the mountain by \( At \) and adds a strip along the base. These strips have increasing length so 

\[
\frac{d}{dt} (\Delta A) > 0.
\]
23 $T_{10} \approx 50,000,000; T_{100} \approx 50,000,000; 25,000\pi$

25 $a = 4, b = 2, c = 1; \int_0^t (4x^2 + 2x + 1)dx = \frac{10}{3}$; Simpson fits parabola

27 $c = \frac{1}{4320}$

2 The trapezoidal error has a factor $(\Delta x)^2$. It is reduced by 4 when $\Delta x$ is cut in half. The error in Simpson’s rule is proportional to $(\Delta x)^4$ and is reduced by 16.

4 Computing $L_n$ and $R_n$ requires $n$ evaluations each. $T_n = \frac{1}{2}y_0 + y_1 + \cdots$ requires $n + 1$: more efficient.

8 The trapezoidal rule for $\int_0^{2\pi} \frac{dr}{3+\sin^2 x} = \frac{\pi}{\sqrt{2}}$ gives $\frac{2\pi}{3} \approx 2.09$ (two intervals), $\frac{T_5}{5} \approx 2.221$ (three intervals), $\frac{17\pi}{24} \approx 2.225$ (four intervals is worse?), and 7 digits for $T_5$. Curious that $M_n = T_n$ for odd $n$.

10 The midpoint rule is exact for 1 and $x$. For $y = x^2$ the integral from 0 to $\Delta x$ is $\frac{1}{3}(\Delta x)^3$ and the rule gives $(\Delta x)^3(\frac{3}{2})^2$. This error $\frac{1}{3}(\Delta x)^3 - \frac{1}{6}(\Delta x)^3 = -\frac{1}{12}(\Delta x)^3$ does equal $\frac{1}{24}(y'(\Delta x) - y'(0))$.

12 The first and third integrals give accurate answers more easily.

14 Correct answer $\frac{2}{3}$. $T_1 = .5, T_{10} \approx .66051, T_{100} \approx .66646$. $M_1 \approx .707, M_{10} \approx .66838, M_{100} \approx .66673$.

What is the rate of decrease of the error?

16 $\int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{dx}{2+\cos 6\pi x} = \frac{2}{\sqrt{3}}$ is approximated by $T_2 = 1(\frac{1}{2} \cdot \frac{1}{3} + \frac{1}{2} \cdot \frac{1}{3}) = \frac{2}{3}$ and

$S_2 = \frac{1}{6}(\frac{1}{2} + 4 \cdot \frac{1}{4} + 2 \cdot \frac{1}{3} + 4 \cdot \frac{1}{2} + 4 \cdot \frac{1}{3}) = \frac{14}{9}$ and $G_1 = \frac{1}{2+\cos(-6\pi/\sqrt{3})} + \frac{1}{2+\cos(6\pi/\sqrt{3})} = .776$ (large error)

and $G_2 = \frac{1}{2+\cos(6\pi \frac{1+\sqrt{2}}{1+\sqrt{2}})} + \frac{1}{2+\cos(6\pi \frac{1-\sqrt{2}}{1-\sqrt{2}})} \approx 1.5$.

18 The trapezoidal rule $T_4 = \frac{\pi}{6}(\frac{1}{2} + \cos 2\pi \frac{1}{3} + \cos 2\pi \frac{2}{3} + \cos 2\pi \frac{3}{3} + 0)$ gives the correct answer $\pi$.

20 $\frac{1}{90}(7y_0 + 32y_1/4 + 12y_1/2 + 32y_3/4 + 7y_1)$ is correct over an interval for $y = 1, x, x^2, x^3, x^4$. Those five requirements give the five coefficients.

22 Any of these stopping points should give the integral as 0.886227 ···. Extra correct digits depend on the computer design.

24 Directly $T_4 \approx 5.4248$. Separately on the intervals $[0, \pi]$ and $[\pi, 4\pi]$, a single trapezoidal step $T_1$ is exact because $|x - \pi|$ is linear. Integral $= \frac{\pi^2}{2} + (8 - 4\pi + \frac{\pi^2}{2})$.

26 Simpson’s rule gives $\frac{1}{6}(0^4 + 4(\frac{1}{2})^4 + 1^4) = \frac{5}{24}$. The difference from $\int_0^1 x^4 dx = \frac{1}{5}$ is $\frac{1}{120}$. Then $y''(1) = 24$ and $y''(0) = 0$ and $\frac{1}{120} = c(24)$ gives $c = \frac{1}{2880}$.

28 $y(a) = y(b)$. 

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