6.1 An Overview

In $10^4 = 10,000$, the exponent 4 is the logarithm of 10,000. The base is $b = 10$. The logarithm of $10^m$ times $10^n$ is $m + n$. The logarithm of $10^m/10^n$ is $m - n$. The logarithm of 10,000 is $4x$. If $y = b^x$ then $x = \log_b y$. Here $x$ is any number, and $y$ is always positive.

A base change gives $b = a^{\log_a b}$ and $b^x = a^{x \log_a b}$. Then $8^x$ is $2^{15}$. In other words $\log_2 y$ is $\log_2 8$ times $\log_8 y$. When $y = 2$ it follows that $\log_2 8$ times $\log_8 2$ equals 1.

On ordinary paper the graph of $y = mx + b$ is a straight line. Its slope is $m$. On semilog paper the graph of $y = Ab^x$ is a straight line. Its slope is $\log b$. On log-log paper the graph of $y = Ax^k$ is a straight line. Its slope is $k$.

The slope of $y = b^x$ is $dy/dx = cb^x$, where $c$ depends on $b$. The number $c$ is the limit as $h \to 0$ of $b^{h} - 1$. Since $x = \log_b y$ is the inverse, $(dx/dy)(dy/dx) = 1$. Knowing $dy/dx = cb^x$ yields $dx/dy = l/cb^x$. Substituting $bx$ for $y$, the slope of $\log_b y$ is $l/cy$. With a change of letters, the slope of $\log_b x$ is $l/cx$.

1 5; -5; -1; 1/2; 1/3; 2 5 1; -10; 80; 1; 4; -1
7 $n \log_b x$ 9 $10^{3}; 10^{4}$
13 $10^{5}$

15 $I_{SF} = 10^5 I_0$; $8.3 + \log_{10} 4$ 17 $A = 7, b = 2.5$ 19 $A = 4, k = 1.5$
21 $a_{x-1} = x_{x+1}^{1/2}; \log 2$ 22 $y - 1 = cx; y - 10 = c(x - 1)$
25 $(x^h - 1)/(-h) = (10^h - 1)/(-h)$ 27 $y'' = cy^2; x'' = -1/cy^2$
29 Logarithm

2 (a) 5 (b) 25 (c) 1 (d) 2 (e) $10^4$ (f) 3
4 The graph of $2^{-x}$ goes through $(0, 1), (1, 1/2), (2, 1/4)$. The mirror image is $x = \log_{1/2} y$ (y is now horizontal):

$\log_{1/2} 2 = -1$ and $\log_{1/2} 4 = -2$.

6 (a) 7 (b) 3 (c) $\sqrt{10}$ (d) $1/4$ (e) $\sqrt{8}$ (f) 5
8 $\log_{b} a = (\log_{a} b)(\log_{d} a)$ and $(\log_{b} d)(\log_{a} c) = \log_{a} c$. Multiply left sides, multiply right sides, cancel $\log_{b} d$.

10 Number of decimal digits $\approx$ logarithm to base 10. For $2^{1000}$ this logarithm is $1000 \log_{10} 2 \approx 1000(0.3) = 300$.
12 $y = \log_{10} x$ is a straight line on “inverse” semilog paper: y axis normal, x axis scaled logarithmically (so $x = 1, 10, 100$ are equally spaced). Any equation $y = \log_{b} x + C$ will have a straight line graph.
14 $y = 10^{1-x}$ drops from 10 to 1 to .1 with slope -1 on semilog paper; $y = 1/2 \sqrt{10^x}$ increases with slope $1/2$ from $y = 5$ at $x = 2$.
16 If 440/second is the frequency of middle A, then the next A is 880/second. The 12 steps from A to A are approximately multiples of $2^{1/12}$. So 7 steps multiplies by $2^{7/12} \approx 1.5$ to give (1.5) (440) = 660. The seventh note from A is E.
18 $\log y = 2 \log x$ is a straight line with slope 2; $\log y = \frac{1}{2} \log x$ has slope $\frac{1}{2}$.
20 $g(f(x)) = y$ gives $g'(f(x)) \frac{dy}{dx} = 1$ or $cg(f(y)) \frac{dy}{dx} = 1$ or $cy \frac{dy}{dx} = 1$ or $\frac{dy}{dx} = \frac{1}{cy}$.
22 The slope of $y = 10^{x}$ is $\frac{dy}{dx} = c10^x$ (later we find that $c = \ln 10$). At $x = 0$ and $x = 1$ the slope is $c$ and $10c$.

So the tangent lines are $y - 1 = c(x - 0)$ and $y - 10 = 10c(x - 1)$. 63
The number \( e \) is approximately 2.718. It is the limit of \((1 + \frac{1}{n})^n \) as \( n \to \infty \). An equivalent form is \( e = \lim_{h \to 0} (1 + h)^{1/h} \). When the base is \( b = e \), the constant \( c \) in Section 6.1 is 1. Therefore the derivative of \( y = \log_b x \) is \( \frac{dy}{dx} = \frac{1}{y} \). The slopes at \( x = 0 \) and \( y = 1 \) are both 0. The notation for \( \log_b y \) is \( \ln y \), which is the natural logarithm of \( y \).

The constant \( e \) in the slope of \( b^x \) is \( e = \ln b \). The function \( b^x \) can be rewritten as \( e^{x \ln b} \). Its derivative is \( (\ln b)e^{x \ln b} = (\ln b)b^x \). The derivative of \( e^x \cdot \sin x \) is \( e^x \cos x - e^x \sin x \). The derivative of \( e^{ax} \) brings down a factor \( a \).

The integral of \( e^x \) is \( e^x + C \). The integral of \( e^{cx} \) is \( \frac{1}{c}e^{cx} + C \). The integral of \( e^{u(x)} dx \) is impossible to find.

\[
\begin{align*}
1 \quad & 49e^{7x} & 3 \quad & 8e^{8x} & 5 \quad & 3^x \ln 3 & 7 \quad & (\frac{2}{3})^x \ln \frac{2}{3} & 9 \quad & -e^{x} & 11 \quad & 2 & 13 \quad & xe^x & 15 \quad & \frac{4}{x^4+e^{-x}}
\end{align*}
\]

24 \( h = 1 \) gives \( c = 9 \); \( h = .1 \) gives \( c = 2.6 \); \( h = .01 \) gives \( c = 2.339 \); \( h = .001 \) gives \( c = 2.305 \); the limit is \( c = \ln 10 = 2.3026 \).

26 (The right base is \( b = e \).) With \( h = \frac{1}{10} \) we pick the base so that \( \frac{1}{10}^{\frac{1}{10}} = 1 \) or \( b^{\frac{1}{10}} = (1 + \frac{1}{10})^{\frac{1}{10}} \)

or \( b = (1 + \frac{1}{10})^{\frac{10}{10}} = \frac{693}{265} \). Generally \( b = (1 + h)^{1/h} \) which approaches \( e \) as \( h \to 0 \).

28 \( h \to 0 \)

\[
\lim_{h \to 0} \frac{\ln \frac{10}{h}}{h} = \lim_{h \to 0} \frac{\ln \frac{10}{h}}{2h} = \frac{1}{2} \lim_{h \to 0} \frac{10^h - 1}{h} = \frac{1}{2} C.
\]

6.2 The Exponential \( e^x \) (page 241)

The number \( e \) is approximately 2.718. It is the limit of \((1 + h)^{1/h} \) as \( h \to 0 \). This gives 1.01^{100} when \( h = .01 \). An equivalent form is \( e = \lim_{(1 + \frac{1}{n})^{n}} \).

When the base is \( b = e \), the constant \( c \) in Section 6.1 is 1. Therefore the derivative of \( y = e^x \) is \( \frac{dy}{dx} = e^x \).

The derivative of \( x = \log_b y \) is \( \frac{dx}{dy} = \frac{1}{y} \). The slopes at \( x = 0 \) and \( y = 1 \) are both 0. The notation for \( \log_b y \) is \( \ln y \), which is the natural logarithm of \( y \).

The constant \( e \) in the slope of \( b^x \) is \( e = \ln b \). The function \( b^x \) can be rewritten as \( e^{x \ln b} \). Its derivative is \( (\ln b)e^{x \ln b} = (\ln b)b^x \). The derivative of \( e^{u(x)} \) is \( e^{u(x)} \frac{du}{dx} \). The derivative of \( \sin x \) is \( e^{\ln x} \cos x \). The derivative of \( e^{Cx} \) brings down a factor \( c \).

The integral of \( e^x \) is \( e^x + C \). The integral of \( e^{cx} \) is \( \frac{1}{c}e^{cx} + C \). The integral of \( e^{u(x)} dx \) is \( \frac{4}{x^4+e^{-x}} \). In general the integral of \( e^{u(x)} \) by itself is impossible to find.

\[
\begin{align*}
1 \quad & 49e^{7x} & 3 \quad & 8e^{8x} & 5 \quad & 3^x \ln 3 & 7 \quad & (\frac{2}{3})^x \ln \frac{2}{3} & 9 \quad & -e^{x} & 11 \quad & 2 & 13 \quad & xe^x & 15 \quad & \frac{4}{x^4+e^{-x}}
\end{align*}
\]
24 \( y = e^{-x} \) solves \( \frac{dy}{dx} = -y \). The difference equation \( Y(x + \frac{1}{4}) = Y(x) - \frac{1}{4}Y(x) \) with \( Y(0) = 1 \) gives \( Y(\frac{1}{4}) = \left(\frac{3}{4}\right)^4 \). (Compare \( e^{-1} \approx 0.37 \) with \( \left(\frac{3}{4}\right)^4 \approx 0.32 \). See the end of Section 6.6.)

26 \( \sqrt{x^2 + 1} \) is the same as \( x^2 \). Its graph at \( x = -2, 0, 2 \) has the same heights \( 1, 0, 1, 3 \) as the graph of \( e^x \) at \( x = -1, 0, 1 \).

28 \( (e^x)'(e^y)' = e^{x+y} \) which is the derivative of \( \frac{1}{2} \).

30 \( e^{-x} \) has antiderivative \( \frac{-1}{2} e^{-x} \ln 2 = \frac{-1}{2} e^{-x} \).

32 \( e^{-x} + x^{-2} \) has antiderivative \( -e^{-x} + \frac{1}{x^2} \).

34 \( e \) meets \( x^2 \) at \( x = 1 \). Their slopes are \( e \) and \( 2x \) by Example 6. At \( x = 1 \) those slopes are \( e \) and \( 2x \).

36 \( x^2 - e^x \) has zero at \( x = 0 \) and \( x = 2 \). The second derivative is negative so the maximum \( (x^2 - e^x)' \) is \( x = 2 \). Check: \( (x^2 - e^x)'|_{x=2} = 3 e^{2} \).

38 \( x^2 - e^x \) increases to \( 1 \) at \( x = 2 \) and then decreases; it never equals \( 1 \) again.

40 \( \frac{\ln 2}{x} \) is the same as \( \frac{\ln 2}{x} \). Its graph at \( x = -2, 0, 2 \) has the same heights \( 1, 0, 1, 3 \) as the graph of \( e^x \) at \( x = -1, 0, 1 \).

42 \( \frac{\ln x}{x} \) has slope \( \frac{e^{\ln x}}{x} (\ln x + 1) = \frac{e^{\ln x}}{x} (1 + \ln x) \).

44 \( x \) meets \( e^{-x} \) at \( x = 1 \). This is the only point where \( x e^{-x} = 1 \) because the derivative is \( x e^{-x} (1 - x) + x e^{-x} = \frac{x}{x} \).

46 \( \ln x = e^{\ln x} e^{-x} = -e^{-1} + e \).

48 \( \frac{\ln x}{x} \) is the same as \( \frac{\ln x}{x} \). Its graph at \( x = -2, 0, 2 \) has the same heights \( 1, 0, 1, 3 \) as the graph of \( e^x \) at \( x = -1, 0, 1 \).

52 \( \int_0^\infty x e^{-x} dx = \frac{1}{2} \) and \( \int_0^\infty x^2 e^{-x} dx = \frac{1}{2} \).

54 \( \int_0^1 (1 - e^{-x})^2 dx = \frac{1}{11} \).

56 \( y(x) = 5y(x) \) is solved by \( y = Ae^{5x} \). The maximum of \( x^5 - x^6 \) occurs when its derivative \( (6x^5 - 6x^6) \) is zero. Then \( x = 6 \) (note that \( z = 0 \) is a minimum).

62 \( \lim_{z \to 0} \frac{e^z}{z} = \lim_{z \to 0} \frac{\ln z}{z} = 1 \).

64 \( e^x - x = 2 \).

66 \( \frac{\ln 2}{x} \) is the same as \( \frac{\ln 2}{x} \). Its graph at \( x = -2, 0, 2 \) has the same heights \( 1, 0, 1, 3 \) as the graph of \( e^x \) at \( x = -1, 0, 1 \).

68 \( \frac{\ln 2}{x} \) is the same as \( \frac{\ln 2}{x} \). Its graph at \( x = -2, 0, 2 \) has the same heights \( 1, 0, 1, 3 \) as the graph of \( e^x \) at \( x = -1, 0, 1 \).

6.3 Growth and Decay in Science and Economics (page 250)

If \( y' = cy \) then \( y(t) = y_0 e^{ct} \). If \( dy/dt = 7y \) and \( y_0 = 4 \) then \( y(t) = 4 e^{7t} \). This solution reaches \( 8 \) at \( t = \frac{\ln 2}{7} \).

The constant solution to \( dy/dt = y + 6 \) is \( y = -6 \). The general solution is \( y = Ae^t - 6 \). If \( y_0 = 4 \) then \( A = 10 \).

The solution of \( dy/dt = cy + s \) starting from \( y_0 \) is \( y = Ae^{ct} + B = (y_0 + \frac{3}{c}) e^{ct} - \frac{3}{c} \). The output from the source is \( \frac{3}{c} (e^{ct} - 1) \). An input at time \( T \) grows by the factor \( e^{c(t-T)} \) at time \( t \).

At \( c = 10\% \), the interest in time \( dt \) is \( dy = 0.01 \ y \ dt \). This equation yields \( y(t) = y_0 e^{0.1t} \). With a source term instead of \( y_0 \), a continuous deposit of \( s = 4000/\text{year} \) yields \( y = 40,000(e - 1) \) after ten years. The deposit
required to produce 10,000 in 10 years is \( s = \frac{yc}{(e^t - 1)} = 1000/(e - 1) \). An income of 4000/year forever comes from \( \frac{yc}{e} = 40,000 \). The deposit to give 4000/year for 20 years is \( \frac{yc}{1 - e^{-2}} \). The payment rate \( s \) to clear a loan of 10,000 in 10 years is \( \frac{1000e}{(e - 1)} \) per year.

The solution to \( y' = -3y + s \) approaches \( y = \frac{s}{3} \).

\[
1. t^2 + y_0 \\
3. ye^{2t} \\
5. 10 e^{4t}; t = \ln 10 \\
7. \frac{1}{4} e^{4t} + 9.75; t = \ln \frac{361}{4} \\
11. c = \ln 2; t = \ln \frac{10}{c} \\
13. \frac{558}{7} \ln \left( \frac{5}{7} \right) \\
15. c = \ln 20; t = \frac{1}{t} \ln \left( \frac{20}{3} \right) \\
17. t = \frac{ln(1/240)}{ln(1.96)} \\
19. e^c = 3 \text{ so } y_0 = e^{-3c}1000 = \frac{1000}{27} \\
21. p = 1013 e^{ch}; 50 = 1013 e^{20c}; c = \frac{1}{20} ln(\frac{50}{1013}); p(10) = 1013 e^{10c} = 1013 \sqrt{\frac{50}{1013}} = \sqrt{1013(50)} \\
25. c = \frac{ln3}{2}; (\frac{1}{2})^3 = \frac{1}{8} \\
29. y = y_0 - at \text{ reaches } y_1 \text{ at } t = \frac{ln a}{a}; \text{ then } y = Ae^{-at} + y_1 \\
27. \text{F/ F; T; T} \\
35. 6; 6 + Ae^{-2t}; 6 - 6e^{-2t}; 6 + 4e^{-2t}; 6 \\
39. ye^{-t}; y(t) = te^t \\
41. A = 1, B = -1 \text{ C = -1} \\
43. e^{0.725} > .075 \\
45. s(e - 1); s(e - 1) \\
47. (1.02)(1.03) \rightarrow 5.06\%; 5\% by Problem 27 \\
49. 20,000 e^{(20-T)(-5)} = 34,400 \text{ (it grows for } 20 - T \text{ years}) \\
51. s = -cyo e^{ct - 1} = -(.01)(1000) e^{60}/(e^{60} - 1) \\
53. y_0 = 1000(.1 - e^{-.005t(48)})) \\
55. e^{4c} = 1.20 so c = \ln 1.20 \text{ } 57. 24e^{3.65} = ? \\
59. \text{T = -oo; constant; to } + \text{ oo} \\
61. \frac{dy}{dt} = 60e^t; \frac{dy}{dt} = 60( -Y + 5); \text{ still } Y_{oo} = 5 \\
63. y = 60e^{cd} + 20, 60 = 60e^{12c} + 20, c = \frac{12}{5} \ln \left( \frac{50}{60} \right) \\
100 = 60e^{cd} + 20 \text{ at } t = \frac{1}{3} \ln \left( \frac{50}{60} \right) \\
56. 60 \\
66. 10 \\
14. Following Example 2, the ratio \( e^{cd} \) would be 90\% or .9. Then \( t = \ln 9 \) and \( c = \ln 10 \approx .23 \). 
16. \( 8e^{0.1t} = 6e^{0.14t} \) gives \( \frac{e}{5} = e^{0.04t} \) and \( t = \frac{1}{0.04} \ln \frac{5}{6} = 250 \ln \frac{5}{6} = 72 \text{ years}. \\
18. \text{At } t = 3 \text{ days, } e^{3c} = 40\% = .4 \text{ and } c = \frac{1}{3} \ln 1.4 = - .3. \text{ At } T \text{ days, } 20\% \text{ remember: } e^{-0.3T} = 20\% = .2 \text{ at } T = \frac{\ln 0.2}{-0.3} = 5.36 \text{ days. (Check after 6 days: )} \text{ (.4)^2 = 16\% will remember.)} \\
20. If y is divided by 10 in 4 time units, it will be divided by 10 again in 4 more units. Thus y = 1 at t = 12. Returning to t = 0 multiples by 10 so yo = 1000. 
22. Exponential decay is y = Ae^{ct}. Then y(0) = A and y(2t) = Ae^{2ct}. The square root of y(0)y(2t) = A^2e^{2ct} is y(t) = Ae^{ct}. One way to find y(3t) = Ae^{3ct} is y(0) = y(0)(y(2t)/y(0))^{3/2}. (A better question is to find y(4t) = Ae^{4ct} = y(0)(y(2t)/y(0))^2 = (y(2t)/y(0))^3.)
24 Go from 4 mg back down to 1 mg in \( T \) hours. Then \( e^{-0.01T} = \frac{1}{4} \) and \(-0.01T = \ln \frac{1}{4} = T = \frac{\ln 4}{-0.01} = 139 \) hours (not so realistic).

26 The second-order equation is \( \frac{d^2 y}{dt^2} - c \left( \frac{dy}{dt} - C \right) y = \frac{d^2 y}{dt^2} - (c + C) \frac{dy}{dt} + C y = 0 \). Check the solution \( y = Ae^{ct} + Be^{Ct} \) by substituting into the equation: \( c^2 A e^{ct} + C^2 B e^{Ct} - (c + C)(c A e^{ct} + C B e^{Ct}) + c C (A e^{ct} + B e^{Ct}) \) does equal zero.

28 Given \( m u = m v - v \Delta m + m \Delta v - (\Delta m) \Delta v \) cancel terms to leave \( m \Delta v - (\Delta m) \Delta v = 7 \Delta m; \) divide by \( \Delta m \) and approach the limit \( m \frac{dV}{dm} = 7 \). Then \( v = 7 \ln m + C \). At \( t = 0 \) this is \( 20 = 7 \ln 4 + C \) so that \( v = 7 \ln m + 20 - 7 \ln 4 = 7 \ln \frac{m}{4} + 20 \).

30 Substitute \( y = Ae^{-t} + B \) into \( y' = 8 - y \) to find \(-Ae^{-t} = 8 - Ae^{-t} - B \). Then \( B = 8 \). At the start \( y_0 = A + B = A + 8 \) so \( A = y_0 - 8 \). Then \( y = (y_0 - 8)e^{-t} + 8 \) or \( y = y_0 e^{-t} + 8(1 - e^{-t}) \).

32 Apply formula (8) to \( \frac{dV}{dm} = -y - 1 \) with \( y_0 = 0 \). Then \( y(t) = \frac{1}{c}(e^t - 1) = 1 - e^t \).

34 Formula (8) applied to \( \frac{dV}{dt} = -y - 1 \) with \( y_0 = 0 \) gives \( y = \frac{-1}{c}(e^{-t} - 1) = e^{-t} - 1 \).

36 (a) \( \frac{dV}{dt} = 3y + 6 \) gives \( y \to \infty \) \( \) (b) \( \frac{dV}{dt} = -3y + 6 \) gives \( y \to 2 \) \( \) (c) \( \frac{dV}{dt} = -3y - 6 \) gives \( y \to -\infty \).

38 Solve \( y' = y + e^t \) by adding inputs at all times \( T \) times growth factors \( e^{t-T} \): \( y(t) = \int_0^t e^{t-T} e^t dT = \int_0^t e^t e^t dT = te^t \). Substitute in the equation to check: \( (te^t)' = te^t + e^t \).

40 Solve \( y' + y = 1 \) by multiplying to give \( e^t y' + ey = e^t \). The left side is the derivative of \( ye^t \) (by the product rule). Integrate both sides: \( ye^t - y_0 e^0 = e^t - e^0 \) or \( ye^t = y_0 e^{-t} + 1 - e^{-t} \).

42 $1000 changes by \$ (1000) (0.04dt), a decrease of 40dt dollars in time dt. The printing rate should be \( s = 40 \).

44 First answer: With continuous interest at \( c \). The multiplier after a year is \( e^{0.09} = 1.094 \) and the effective rate is 9.4%. Second answer: The continuous rate \( c \) that gives an effective annual rate of 9% is \( c = 0.09 \) or \( c = \ln 1.09 = .086 \) or 8.6%.

46 \( y_0 \) grows to \( y_0 e^{(1/20)} = 50,000 \), so the grandparent gives \( y_0 = 50,000e^{-2} \approx \$ 8767 \). A continuous deposit \( s \) grows to \( \frac{1}{10}(e^{(1/20)} - 1) = 50,000 \), so the parent deposits \( s = \frac{1}{10}(50,000) \approx \$ 783 \) per year.

Saving \( s = 1000/y \) grows to \( 1000/1(e^{t} - 1) = 50,000 \) when \( e^{-1} = 1 + \frac{5000}{1000} \) or \( t = \ln 6 = 0.85 \) or 17.9 years.

48 The deposit of \( 4dT \) grows with factor \( c \) from time \( T \) to time \( t \), and reaches \( e^{ct}dT \). With \( t = 2 \) add deposits from \( T = 0 \) to \( T = 1 \) : \( \int_0^1 e^{(2-t)}dT = \left( \frac{e^{(2-t)}1}{c} \right)_0^1 = \frac{4e^2 - 4e^0}{c} \). \( y(t) = (5000 - \frac{5000}{1000}e^{0.08t} + \frac{5000}{1000}e^{0.08t}) \) is zero when \( e^{0.08t} = \frac{5000 - 500}{5000 - 500} \approx 5 \). Then \( .08t = \ln 5 \) and \( t = \frac{\ln 5}{.08} \approx 20 \) years. (Remember the deposit grows until it is withdrawn.)

52 After 365 days the value is \( y = \frac{e^{(0.1)365}}{e^{.05}} = e^{3.65} = \$ 38. \)

54 (a) Income = expense when \( Io e^{ct} = Eo e^{ct} \) or \( e^{ct} = \frac{Io}{Eo} \) or \( t = \ln \left( \frac{Io}{Eo} \right) \). (b) Integrate \( Eo e^{ct} - Io e^{ct} \) until \( e^{ct} = \frac{Eo}{Io} \). At the upper limit the integral is \( Eo - Io e^{2ct} = \frac{1}{c} \left( \frac{Eo}{Io} - Io \right) \approx \frac{Eo}{2cIo} \). Lower limit is \( t = 0 \) so subtract \( \frac{Eo}{c} - \frac{Io}{c} \) : Borrow \( \frac{Eo}{2cIo} - \frac{Eo}{c} + \frac{Io}{2c} \).

56 After 10 years (halfway through the mortgage) the variable rate \( .09 + .001(10) \) equals the fixed rate 10% = .1. Since the variable was lower early, and therefore longer, the variable rate is preferred.

58 If \( \frac{dy}{dt} = -y + 7 \) then \( \frac{dy}{dt} \) is zero at \( y = 7 \). Then \( \frac{dy}{dt} = 0 \) gives \( \frac{dy}{dt} \), and \( y = 7 - e^{-t}(y_0 - 7) \). The decay rate is \( c = -1 \), and \( y = 7 - e^{-t}(y_0 - 7) \).

60 All solutions to \( \frac{dy}{dt} = c(y - 12) \) converge to \( y = 12 \) provided \( c \) is negative.

62 (a) False because \( (y_1 + y_2)' = cy_1 + sy_2 + cy_2 + s \). We have \( 2s \) not \( s \). (b) True because \( (\frac{1}{2} y_1 + \frac{1}{2} y_2)' = \frac{1}{2} cy_1 + \frac{1}{2} cy_2 + \frac{1}{2} cy_2 + \frac{1}{2} s \). (c) False because the derivative of \( y' = cy \) is \( y'' = c(y') \) and \( s \) is gone.

64 The solution is \( y = Ae^{ct} + B \). Substitute \( t = 0, 1, 2 \) and move \( B \) to the left side: \( 100 - B = A, 90 - B = Ae^c, 84 - B = Ae^{2e} \). Then \( 100 - B)(84 - B) = (90 - B)(90 - B) \); both sides are \( A^2 e^{2c} \).

Solve for \( B = 8400 - 184B + B^2 = 8100 - 180B + B^2 \) or \( 300 = 4B \). The steady state is \( B = 75 \). (This problem is a good challenge and was meant to have a star.)
6.4 Logarithms (page 258)

The natural logarithm of \( x \) is 
\[
    \ln x = \int_{1}^{x} \frac{dt}{t}
\]  
(Or \( \frac{d}{dx} \ln x \).) This definition leads to \( \ln(xy) = \ln x + \ln y \) and \( \ln z^n = n \ln x \). Then \( e \) is the number whose logarithm (area under \( 1/x \) curve) is 1. Similarly \( e^x \) is now defined as the number whose natural logarithm is \( x \). As \( x \to \infty \), \( \ln x \) approaches infinity. But the ratio \( (\ln x)/\sqrt{x} \) approaches zero. The domain and range of \( \ln x \) are \( 0 < x < \infty \), \( -\infty < \ln x < \infty \).

The derivative of \( \ln x \) is \( \frac{1}{x} \). The derivative of \( \ln(1 + x) \) is \( \frac{1}{1 + x} \). The tangent approximation to \( \ln(1 + x) \) at \( x = 0 \) is \( x \). The quadratic approximation is \( x - \frac{1}{2}x^2 \). The quadratic approximation to \( e^x \) is \( 1 + x + \frac{1}{2}x^2 \).

The derivative of \( \ln u(x) \) by the chain rule is \( \frac{u'(x)}{u(x)} \). Thus \( \ln(\cos x)' = -\frac{\sin x}{\cos x} = -\tan x \). An antiderivative of \( \tan x \) is \( -\ln \cos x \). The product \( p = xe^{x^2} \) has \( \ln p = 5x + \ln x \). The derivative of this equation is \( p'/p = 5 + \frac{1}{x} \). Multiplying by \( p \) gives \( p' = xe^{5x}(5 + \frac{1}{x}) = 5xe^{5x} + e^{5x} \), which is LD or logarithmic differentiation.

The integral of \( u'(x)/u(x) \) is \( \ln u(x) \). The integral of \( 2x/(x^2 + 4) \) is \( \ln(x^2 + 4) \). The integral of \( 1/cx \) is \( \frac{\ln x}{c} \). The integral of \( 1/(ct + s) \) is \( \ln(ct + s) \). The integral of \( 1/\cos x \), after a trick, is \( \ln(\sec x + \tan x) \). We should write \( \ln |x| \) for the antiderivative of \( 1/x \), since this allows \( x < 0 \). Similarly \( \int du/u \) should be written \( \ln|u| \).

1. \( \frac{1}{2} \) 3. \( \frac{1}{2} \) 5. \( \cos x = \cot x \) 7. \( \frac{x}{2} \) 9. \( \frac{1}{2} \) 11. \( \frac{1}{3} \ln t + C \) 13. \( \ln \frac{4}{3} \)

6. \( 1/2 \) 8. \( \ln x \) 10. \( \frac{1}{x} \) 12. \( \ln(1 + x) \) from \( \frac{du}{u} \). 14. \( \frac{1}{2} \ln(3 + 2t) \) 15. \( \frac{1}{2} \ln(5 - 3t) \) 16. \( z^2 + 1 \) 18. \( \int \frac{du}{u^2} = \frac{1}{u} + \frac{1}{u^2} \)
20 $\int \frac{\sin x}{\cos x} \, dx = \int -\frac{du}{u} = -\ln u = -\ln(\cos x)_{0}^{\pi/4} = -\ln \frac{1}{\sqrt{2}} + 0 = \frac{1}{2}\ln 2.$

22 $\int \frac{\cos 3x}{\sin 3x} \, dx = \frac{1}{3}\ln(\sin 3x) + C.$

24 Set $u = \ln \ln x$. By the chain rule $\frac{du}{dx} = \frac{1}{x}$. Our integral is $\int \frac{du}{u} = \ln u = \ln(\ln(\ln x)) + C.$

26 The graph starts at $-\infty$ when $x = 0$. It reaches zero when $x = \frac{5}{3}$ and goes down again. At $x = \pi$ it stops.

28 $\ln y = \frac{1}{2}\ln(x^{2} + 1) + \frac{1}{2}\ln(x^{2} - 1)$. Then $\frac{dy}{dx} = \frac{x^{2} + 1}{2x^{2}} + \frac{x^{2} - 1}{2(x^{2} - 1)} = \frac{3x^{2}}{x^{4} - 1}$. Then $\frac{dy}{dx} = \frac{3x^{2}}{x^{4} - 1}.$

30 $\ln y = \frac{1}{2}\ln x$ and $\frac{dy}{dx} = \frac{ln x}{x^{2}}$ so $\frac{dy}{dx} = (\ln x)(x^{-1})/x.$

32 $\ln y = e \ln x$ and $\frac{dy}{dx} = e \ln x$ so $\frac{dy}{dx} = e \ln x$. Alternatively we have $y = \frac{1}{x}$ and $\frac{dy}{dx} = -\frac{1}{x^{2}}$.

34 $\ln y = \frac{1}{3}\ln x + \frac{1}{6}\ln x + \frac{1}{6}$, or $\frac{dy}{dx} = \frac{e^{-\ln x}}{x}$. Alternatively we have $\frac{dy}{dx} = \frac{1}{x^{2}}$.

36 $\ln y = -\ln x$ so $\frac{1}{x} = \frac{1}{y}$ and $\frac{dy}{dx} = -e^{-\ln x}$. Alternatively we have $y = \frac{1}{x}$ and $\frac{dy}{dx} = -\frac{1}{x^{2}}$.

38 $\ln x_{1}^{\alpha} + (\ln x_{1})^{-\frac{1}{2}} = (\pi - 0) + (0 - \ln(2 - 2)) = \pi - \ln 2.$

40 $\frac{dy}{dx} = \frac{1}{x}$. Alternatively use $\frac{dy}{dx} = (x^{2}) - \frac{1}{x^{2}} (x) = \frac{1}{x}$.

42 This is $\int \frac{dx}{x}$ with $u = \sec x + \tan x$ so the integral is $\ln(\sec x + \tan x)$. See Problem 41!

44 $\frac{dy}{dx}(\ln(x - a) - \ln(x + a)) = \frac{1}{x-a} - \frac{1}{x+a} = \frac{(x+a)-(x-a)}{(x-a)(x+a)} = \frac{2a}{x^{2} - a^{2}}$.

46 Misprint! $\frac{1 + x^{2}}{\sqrt{2} + x^{2}} \cdot \frac{1 + y^{2}}{\sqrt{2} + y^{2}} = \frac{1 + x^{2} + y^{2} + x^{2} + y^{2}}{\sqrt{2} + x^{2} + y^{2} + x^{2} + y^{2}}$.

48 Linear: $e^{x} \approx 1 + .1 = 1.1$. Quadratic: $e^{x} \approx 1 + .1 + \frac{1}{2}(1.1)^{2} = 1.105$. Calculator: $e^{1} = 1.105170918.$

50 Linear: $e^{x} \approx 1 + 2 + \frac{1}{2}(2^{2}) = 5$. Calculator: $e^{2} = 7.389.$

52 Use l'Hôpital's Rule: $\lim_{x \to 0} \frac{e^{x}}{x}$.

54 Use l'Hôpital's Rule: $\lim_{x \to 0} \frac{1}{b^{x} \ln b} = \ln b$. We have redefined the derivative of $b^{x}$ at $x = 0$.

56 Upper rectangles $\frac{1}{4} + \frac{1}{6} + \frac{1}{6} + \frac{1}{6} \approx .7595$ Lower rectangles: $\frac{1}{2} + \frac{1}{6} + \frac{1}{3} + \frac{1}{3} \approx .6345$. Exact area in $2 \approx .693$.

58 $\frac{1}{2}$ is smaller than $\frac{1}{2} \sqrt{2}$ when $1 < t < 2$. Therefore $\int_{1}^{a} dt < \int_{1}^{a} \frac{dt}{\sqrt{2}}$ or $\ln x < 2\sqrt{2} - 2$. (In Problem 59 this leads to $\ln x \to 0$. Another approach is from $\frac{2}{e} \to 0$ in Problem 6.25.) If $x$ is much smaller than $e^{x}$ then $\ln x$ is much smaller than $x$.

60 From $\frac{\ln x}{x} \to 0$ we know $\frac{\ln x}{x^{1/2}} \to 0$. This is $\frac{\ln x}{x^{1/2}} \to 0$. Since $x$ is fixed we have $\frac{\ln x}{x^{1/2}} \to 0$.

62 $\frac{1}{x} \ln \frac{1}{x} = \frac{\ln x}{x} \to 0$ as $x \to \infty$. This means $\ln y \to 0$ as $y = \frac{1}{x} \to 0$. (Emphasize: The factor $y \to 0$ is "stronger" than the factor $\ln x \to -\infty$.)

64 From $\int_{1}^{x} e^{t-1}dt = \frac{e^{x} - 1}{h}$ we find $\int_{1}^{x} t^{-1}dt = \lim_{h \to 0} \frac{e^{x} - 1}{h}$. The left side is recognized as $\ln x$. (The right side is the "mysterious constant $c$" when the base is $b = e$. We discovered earlier that $c = \ln b$.)

66 .01 - $\frac{1}{2} (0.01)^{2} + \frac{1}{2} (0.01)^{3} = .00995033$. Also $\ln 1.02 \approx .02 - \frac{1}{2} (0.02)^{2} + \frac{1}{6} (0.02)^{3} = .13786267$.

68 To emphasize: If the ant didn't crawl, the fraction $y$ would be constant (the ant would move as the band stretches). By crawling $v dt$ the fraction $y$ increases by $\frac{x dt}{\text{band length}}$. So $\frac{dy}{dt} = \frac{y}{t} = \frac{1}{t^{2} + 2}$. Then $y = \frac{1}{2} \ln(8t + 2) + \frac{1}{2}(\ln(8t + 2) - \ln 2)$. This equals 1 when $8 = \ln 8t + 2$ or $4t + 1 = e^{8}$ or $t = \frac{1}{4}(e^{8} - 1)$

70 LD: $\ln p = \ln x = \ln x$ so $\frac{dp}{dx} = 1 + \ln x$ and $\frac{dy}{dx} = p(1 + \ln x) = x^{2}(1 + \ln x)$. None find the same answer by ED: $\frac{dy}{dx} = x^{2}(1 + \ln x)$.

72 To compute $\int_{1}^{2} \frac{dx}{x}$ with error $\approx 10^{-5}$ the trapezoidal rule needs $\Delta x \approx 10^{-2}$, six Simpson steps:

$S_{6} = \frac{1}{36} \left( \frac{1}{1} + \frac{2}{13} + \frac{2}{13} + \frac{2}{13} + \frac{2}{13} + \frac{2}{13} \right) \approx \frac{1}{36}(1 + 2 + 2 + 2 + 2 + 2) = 0.693149$ compared to $\ln 2 = 0.693147$. Predicted error $\approx \frac{1}{120}(4 - \frac{2}{5}) = 1.6 \times 10^{-6}$, actual error 1.5 $\times 10^{-6}$.

74 $\frac{1}{\ln 90,000} = .0277$ says that about 877 of the next 10,000 numbers are prime: close to the actual count 879.

76 $\ln x^{2} = 2 \ln x^{1/2}$. This equals $\ln y = (t+1) \ln(1+t) = t \ln(1+t)$. The curve $x^{2} = y^{2}$ is asymptotic to $x = 1$, for $t$ near zero. It approaches $x = e$, $y = e$ as $t \to \infty$. It is symmetric across the 45°
6.5 Separable Equations Including the Logistic Equation (page 266)

The equations $dy/dt = cy$ and $dy/dt = cy + s$ and $dy/dt = u(y)v(t)$ are called separable because we can separate $y$ from $t$. Integration of $\int dy/y = \int c \, dt$ gives $\ln y = ct + \text{constant}$. Integration of $\int dy/(y + s/c) = \int c \, dt$ gives $\ln(y + \frac{s}{c}) = ct + C$. The equation $dy/dx = -z/y$ leads to $\int y \, dy = -\int x \, dx$. Then $y^2 + x^2 = \text{constant}$ and the solution stays on a circle.

The logistic equation is $dy/dt = cy - by^2$. The new term $-by^2$ represents competition when $cy$ represents growth. Separation gives $\int dy/(cy - by^2) = \int dt$, and the $y$-integral is $1/c \ln \frac{X}{e - by}$. Substituting $y_0$ at $t = 0$ and taking exponentials produces $y/(c - by) = e^{ct}y_0/(c - by_0)$. As $t \to \infty$, $y$ approaches $\frac{c}{b}$. That is the steady state where $cy - by^2 = 0$. The graph of $y$ looks like an $S$, because it has an inflection point at $\frac{1}{2} \frac{c}{b}$.

In biology and chemistry, concentrations $y$ and $z$ react at a rate proportional to $y$ times $z$. This is the Law of Mass Action. In a model equation $dy/dt = c(y)y$, the rate $c$ depends on $y$. The MM equation is $dy/dt = -cy/(y + K)$. Separating variables yields $\int \frac{y}{y + K} \, dy = \int -c \, dt = -ct + C$.

1 $7e^t - 5 \quad 3 \left(\frac{3}{2}x^2 + 1\right)^{1/3} \quad 5x \quad 7 \, e^{1 - \cos t} \quad 9 \left(\frac{x}{3} + \sqrt{y_0}\right)^2 \quad 11 \, y_\infty = 0; t = \frac{1}{by_0}$

15 $x = 1 + e^{-t}, y$ is in 13 $17 \, ct = \ln 3, ct = \ln 9$

19 $b = 10^{-9}, c = 13 \cdot 10^{-3}; y_\infty = 13 \cdot 10^6$; at $y = \frac{c}{2b}$ (10) gives $\ln \frac{1}{b} = ct + \ln \frac{10^6}{c - 10^6}$ so $t = 1900 + \ln 12 = 2091$

21 $y^2$ dips down and up (a valley) 23 $sc = 1 = sbr$ so $s = \frac{1}{r}, r = \frac{1}{2}$

25 $y = \frac{N}{1 + e^{-N/(N-1)}}; T = \ln(N/(N-1)) \to 0$ 27 Dividing $cy$ by $y + K > 1$ slows down $y'$

29 $\frac{dy}{dt} = \frac{cK}{y + K} (y > 0, \frac{cy}{y + K} \to c$

31 $\frac{dy}{dt} = \frac{y}{y + 1};$ multiply $e^{y/K} \frac{K}{K} = e^{-ct/K} e^{y_0/K}(y_0/K)$ by $K$ and take the $K$th power to reach (19)

33 $y' = (3 - y)^2; y = \frac{1}{3} y + \frac{1}{3} y + \frac{1}{3};$ at $t = \frac{1}{3}$

35 $Ae^t = D = Ae^t + B + Dt + t \to D = -1, B = -1; y_0 = A + B$ gives $A = 1$

37 $y \to 1$ from $y_0 > 0, y \to -\infty$ from $y_0 < 0, y \to 1$ from $y_0 > 0, y \to -1$ from $y_0 < 0$

39 $\int \frac{\cos y}{\sin y} \, dy = \int dt \ln(\sin y) = t + C = t + \ln \frac{1}{2}$. Then $\sin y = \frac{1}{2} e^t$ stops at 1 when $t = \ln 2$

2 $y \, dy = dt$ gives $\frac{1}{2} y^2 = t + C$. Then $C = \frac{1}{2}$ at $t = 0$. So $y^2 = 2t + 1$ and $y = \sqrt{2t + 1}$.

4 $\frac{dy}{dt} = dx$ gives $\tan^{-1} y = x + C$. Then $C = 0$ at $x = 0$. So $y = \tan x$.

6 $\frac{dy}{dx} = \cos x \, dx$ gives $\ln(\sin y) = \sin x + C$. Then $C = \ln(\sin 1)$ at $x = 0$. After taking exponentials $\sin y = (\sin 1) e^{\sin x}$. No solution after $\sin y$ reaches 1 (at the point where $(\sin 1) e^{\sin x} = 1$).

8 $e^{dy} = e^{dt} \to e^y = e^{t} + C$. Then $C = e^t - 1$ at $t = 0$. After taking logarithms $y = \ln(e^t + e^{t-1})$.

10 $\frac{d(\ln x)}{dx} = \frac{d(\ln x)}{dx} = n$. Therefore $\ln y = n \ln x + C$. Therefore $y = (x^n)(e^C) = \text{constant times } x^n$.
12 \ y' = by^2 \text{ gives } y^{-2} dy = b \ dt \text{ and } -\frac{1}{y} = bt + C. \text{ Then } C = -\frac{1}{2} \text{ at } t = 0. \text{ Therefore } y = e^{-\frac{1}{2} bt} \text{ which becomes infinite when } bt = \frac{1}{2} \text{ or } t = \frac{1}{2b}.

14 (a) Compare \frac{d}{dt} \left( \frac{y}{e^{-by}} \right) \text{ with } \frac{\ddot{y}}{e^{-by}}. \text{ In the exponent } c = 1. \text{ Then } \frac{d}{dt} = \frac{1}{2} \text{. Thus } y' = y - \frac{1}{2} y^2 \text{ with } y_0 = 1.

(b) For \frac{d}{dt} \left( \frac{y}{e^{-by}} \right) \text{ the exponent gives } c = 3. \text{ Then also } \frac{d}{dt} = \frac{3}{2} \text{. Thus } y' = 3y - 3y^2 \text{ with } y_0 = \frac{1}{3}.

16 Equation (14) is \( x = \frac{1}{c} \left( b + \frac{c}{b+de^{-ct}} \right) \). Turned upside down this is \( y = \frac{c}{b+de^{-ct}} \) with \( d = \frac{c}{b+yo} \).

18 Correction: \( u = \frac{y}{e^{-by}} \). Then \( \frac{du}{dt} = \frac{1}{e^{-by}} \frac{dy}{dt} \). Substitute \( dy = \frac{dt}{e^{by}} \) to obtain \( \frac{du}{dt} = \frac{c}{e^{by}} \). So \( u_{0e^{ct}} \).

20 \ y' = y + y^2 \text{ has } c = 1 \text{ and } b = -1 \text{ with } y_0 = 1. \text{ Then } y(t) = \frac{1}{1+2e^{-t}} \text{ by formula (12). The denominator } 1+2e^{-t} \text{ is zero when } 2e^{-t} = 1 \text{ or } t = \ln 2.

22 If \( u = \frac{1}{y} \) then \( \frac{du}{dt} = -\frac{2u}{y} = -\frac{2(y-y^2)}{y^2} = -2cu + 2b. \text{ The solution is } u = (u_0 - 2b)e^{-2ct} + \frac{2b}{2c}.

Then \( y = \left( \frac{1}{y_0} - \frac{1}{2b} \right) e^{-2ct} + \frac{1}{2b} \right)^{-1/2} \text{ solves the equation } y' = cy - by^3 \text{ with "cubic competition".}

Another S-curve!

24 \ y_0 = rY_0 \text{ and } \frac{dy}{dt} = \frac{du}{dt} \text{ so } (\frac{dy}{dt})_0 = \frac{r}{Y_0}.

26 At the middle of the S-curve \( y = \frac{y_0}{2} \) and \( \frac{dy}{dt} = c(\frac{y_0}{2}) - b(\frac{y_0}{2})^2 = \frac{y_0^2}{4b}. \text{ If } b \text{ and } c \text{ are multiplied by } 10 \text{ then so is this slope } \frac{y_0}{4b}, \text{ which becomes steeper.}

28 If \( y = y_0 + K \) then \( \frac{dy}{dt} = \frac{dK}{dt} \text{ and } y = \frac{dK}{c-d-t}. \text{ At this steady state the maintenance dose replaces the aspirin being eliminated.}

30 The rate \( R = \frac{dy}{dt} \text{ is a decreasing function of } K \text{ because } \frac{dK}{dt} = -\frac{y}{y+c}\).}
6.6 Powers Instead of Exponentials  (page 276)

compounding multiplies by \( e^x \). At \( x = 10\% \) with continuous compounding, \$1\) grows to \( e^{1.08} \approx \$1.105 \) in a year.

The difference equation \( y(t+1) = ay(t) \) yields \( y(t) = a^t y_0 \). The equation \( y(t+1) = ay(t) + s \) is solved by \( y = a^t y_0 + s[1 + a + a^2 + \cdots + a^{t-1}] \). The sum in brackets is \( \frac{1-a^t}{1-a} \) or \( \frac{a^t-1}{a-1} \). When \( a = 1.08 \) and \( y_0 = 0 \), annual deposits of \( s = 1 \) produce \( y = \frac{1.08^t - 1}{0.8} \) after \( t \) years. If \( a = \frac{1}{2} \) and \( y_0 = 0 \), annual deposits of \( s = 6 \) leave \( 12(1 - \frac{1}{2^t}) \) after \( t \) years, approaching \( y_{\infty} = 12 \). The steady equation \( y_{\infty} = ay_{\infty} + s \) gives \( y_{\infty} = s/(1-a) \).

When \( i \) = interest rate per period, the value of \( y_0 = \$1 \) after \( N \) periods is \( y(N) = (1+i)^N \). The deposit to produce \( y(N) = 1 \) is \( y_0 = (1+i)^{-N} \). The value of \( s = \$1 \) deposited after each period grows to \( y(N) = \frac{1}{1-\frac{1}{(1+i)^N-1}} \). The deposit to reach \( y(N) = 1 \) is \( s = \frac{1}{1-\frac{1}{(1+i)^N-1}} \).

Euler’s method replaces \( y' = cy \) by \( \Delta y = cy \Delta t \). Each step multiplies \( y \) by \( 1 + c \Delta t \). Therefore \( y \) at \( t = 1 \) is \( (1+c \Delta t)^\frac{1}{\Delta t} y_0 \), which converges to \( y_0 e^c \) as \( \Delta t \to 0 \). The error is proportional to \( \Delta t \), which is too large for scientific computing.

1. \[
\begin{align*}
1 &- z + \frac{z^2}{2} - \frac{z^3}{6} + \cdots \\
7 &1 + n\left(\frac{-1}{n} + \frac{n(n+1)}{2}\right) \frac{(-1)^2}{2} - 1 + \frac{1}{2} \\
11 &\text{12--0--10--26--0} \\
19 &y(t) = \frac{1}{2}(3^t - 1) \\
27 &-2,-10,-26 \to \infty; -5, -\frac{17}{2}, -\frac{41}{4} \to \infty \\
35 &\frac{1000}{1-(1.1)^{12}} \\
37 &1000(1.1)^{20} = 673 \\
43 &1.01412 \to 1.184 \to \text{Visa charges 18.4\%} \\
4 \quad &y = 1 + 2x + \frac{1}{2}(2x)^2 + \frac{1}{6}(2x)^3 + \cdots \\
10 &\text{Determine each term and multiply by 2 to find the next term.} \\
14 &y(0) = 0, y(1) = 1, y(2) = 3, y(3) = 7 (and \( y(n) = 2^n - 1 \)). \\
18 &y(t) = t (Notice that \( a = 1 \)). \\
22 &y(t) = 5a^t + [\frac{3^{t-1}}{2}]. \\
26 &\text{Ask for } y(0) + 6 = y(0). \\
30 &\text{The equation } -dP(t+1) + b = cP(t) \text{ becomes } -2P(t+1) + 8 = P(t) \text{ or } P(t+1) = -\frac{1}{2}P(t) + 4. \text{ Starting} \\
32 &\text{from } P(0) = 0 \text{ the solution is } P(t) = 4\left[1 - \left(-\frac{1}{2}\right)^t\right] = \frac{8}{3}(1 - \left(-\frac{1}{2}\right)^t) \to \frac{8}{3}. \\
34 &\text{Present value } = \$1,000 (1.1)^{-20} \approx \$148.64. \\
36 &\text{Correction to formulas 5 and 6 on page 273: Change .05n to .05/n. In this problem } n = 12 \text{ and} \\
\quad &N = 6(12) = 72 \text{ months and .05 becomes } .1 \text{ in the loan formula: } s = \$10,000 \cdot (1.1)/12[1 - (1 + \frac{i}{12})^{-72}] \approx \$185. \\
\]
38 Solve $1000 = 8000 \left[ \frac{1}{1-(1.1)^{-n}} \right]$ for $n$. Then $1 - (1.1)^{-n} = 0.8$ or $(1.1)^{-n} = 0.2$. Thus $1.1^n = 5$ and

$$n = \frac{\ln 5}{\ln 1.1} \approx 17 \text{ years}.$$  

40 The interest is $(.05)1000 = 50$ in the first month. You pay $60. So your debt is now $1000 - 60 = 940. Suppose you owe $y(t)$ after month $t$, so $y(0) = 1000$. The next month's interest is $(.05)1000 = 50$. You pay $60. So $y(t+1) = 1.05y(t) - 60$. After 12 months

$$y(12) = (1.05)^{12}1000 - 60 \approx 841.$$  

42 Compounding $n$ times in a year at 100% per year gives $(1 + i)^n$. Its logarithm is $n \ln(1 + i)$. Therefore $(1 + i)^n \approx e^{(1 - 1/2n)} \approx e^{(1 - 1/2n)}$.

44 Use the loan formula with $.09/n$ not $.09n$: payments $s = 80000 - \frac{.09/12}{1 - (1 + \frac{.09/12}{12}} = 643.70$. Then 360 payments equal $231,732.$

6.7 Hyperbolic Functions  \hspace{1cm} (page 280)

Cosh $x = \frac{1}{2}(e^x + e^{-x})$ and sinh $x = \frac{1}{2}(e^x - e^{-x})$ and cosh$^2 x - \sinh^2 x = 1$. Their derivatives are sinh $x$ and cosh $x$ and zero. The point $(x, y) = (\cosh t, \sinh t)$ travels on the hyperbola $x^2 - y^2 = 1$. A cable hangs in the shape of a catenary $y = a \cosh \frac{x}{a}$.

The inverse functions sinh$^{-1} x$ and tanh$^{-1} x$ are equal to $\ln[x + \sqrt{x^2 + 1}]$ and $\frac{1}{2} \ln \frac{1 + x}{1 - x}$. Their derivatives are $1/\sqrt{x^2 + 1}$ and $1/1 - x^2$. So we have two ways to write the antiderivative. The parallel to cosh $x + \sinh x = e^x$ is Euler's formula \( \cos x + i \sin x = e^{ix} \). The formula $\cos x = \frac{1}{2}(e^{ix} + e^{-ix})$ involves imaginary exponents. The parallel formula for sin $x$ is $\frac{1}{2i}(e^{ix} - e^{-ix})$.

1. $e^x, e^{-x}, \frac{e^{2x} - e^{-2x}}{4} = \frac{1}{2} \sinh 2x$  
7. $\sinh nx$  
9. $3 \sinh(3x + 1)$  
11. $-\frac{\sinh x}{\cosh^2 x} = -\tanh x \sech x$

13. $4 \cosh x \sinh x$  
15. $\sqrt{x^2 + 1} \sech \sqrt{x^2 + 1}$  
17. $6 \sinh^5 x \cosh x$

19. $\cosh(\ln x) = \frac{1}{2}(x + \frac{1}{x}) = 1$ at $x = 1$  
21. $\frac{13}{12}, \frac{13}{12}, -\frac{12}{12}, -\frac{12}{12}$  
23. $0, 0, 1, \infty$

25. $\frac{1}{2} \ln(1 + \cosh x)$  
27. $\frac{1}{2} \cosh^3 x$  
29. $\ln(1 + \cosh x)$  
31. $e^x$

33. $\int y \, dx = \int \sinh t \sinh t \, dt; A = \frac{1}{2} \sinh t \cosh t - \int y \, dx; A' = \frac{1}{2}; A = 0$ at $t = 0$ so $A = 0$.  
41. $e^y = x + \sqrt{x^2 + 1}, y = \ln[x + \sqrt{x^2 + 1}]$  
47. $\frac{1}{2} \ln \left[ \frac{x+y}{2} \right]$  
49. sinh$^{-1} x$ (see 41)  
51. $-\sec^{-1} x$

53. $\frac{1}{2} \ln 3; 55. y(x) = \frac{1}{x} \cosh cx; \frac{1}{x} \cosh cxL - \frac{1}{x}$  
57. $y'' = y - 3y^2; \frac{1}{2}(y')^2 = \frac{1}{2}y^2 - y^3$ is satisfied by $y = \frac{1}{2} \sech^2 x$

2. \( \frac{d}{dx} \left( \frac{e^{2x} - e^{-2x}}{2} \right) = \frac{e^{2x} - e^{-2x}}{2} = \sinh x \); \( \frac{d}{dx} \left( \frac{e^{2x} - e^{-2x}}{2} \right) = \frac{e^{2x} + e^{-2x}}{2} = \cosh x \).

4. \( \frac{d}{dx} \left( \frac{\sinh x}{\cosh x} \right) = \frac{1}{(\cosh x)^2} = \frac{1}{(\cosh x)^2} = \sech^2 x \).

6. The factor $\frac{1}{2}$ should be removed from Problem 5 Then the derivative of Problem 5 is $2 \cosh x \sinh x x + 2 \sinh x \cosh x = 2 \sinh x$. Therefore $2 \sinh x \cosh x$ (similar to sin $2x$).

8. $\frac{e^{2x} + 2}{2} \left( \frac{e^{2x} - e^{-2x}}{2} \right)^2 = \frac{1}{4}(2e^{x+y} - 2e^{-x-y}) = \sinh(x+y)$. The $x$ derivative gives $\cosh(x+y) = \cosh x \cosh y + \sinh x \sinh y$.

10. $2x \cosh x^2$  
12. $\sinh(\ln x) = \frac{1}{2}(e^{\ln x} - e^{-\ln x}) = \frac{1}{2}(x - \frac{1}{x})$ with derivative $\frac{1}{2}(1 + \frac{1}{x^2})$.

14. $\cosh^2 x - \sinh^2 x = 1$ with derivative zero.

16. $\frac{1 + \tanh x}{\tanh x} = e^{2x}$ by the equation following (4). Its derivative is $2e^{2x}$. More directly the quotient rule gives
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\[
\frac{(1-\tanh x)^2 \cosh^2 x + (1+\tanh x) \sech^2 x}{(1-\tanh x)^2} = \frac{2 \sech^2 x}{(1-\tanh x)^2} = \frac{2(\cosh^2 x - \sinh^2 x)}{e^{-2x}} = 2e^{2x}.
\]

18 \( \frac{d}{dx} \ln u = \frac{du}{dx} \) \( \frac{u}{u} = \frac{1}{\tanh x \sech x - \sech^2 x \sech x + \tanh x} \). Because of the minus sign we do not get \( \sech x \). The integral of \( \sech x \) is \( \sinh^{-1}(\tanh x) \) + C.

20 \( \sech x = \sqrt{1 - \frac{(3/4)^2}{2}} = \frac{2}{3} \), \( \coth x = \frac{3}{4} \), \( \cosh x = \frac{2}{\sqrt{5}} \), \( \sinh x = \frac{2}{\sqrt{5}} \), \( \tanh x = \frac{3}{4} \), \( \cosh x = \frac{5}{3} \), \( \cosh x = \frac{4}{3} \).

22 \( \cosh x = \sqrt{(2)^2 + 1} = \sqrt{5} \), \( \tanh x = \frac{2}{\sqrt{5}} \), \( \cosh x = \frac{1}{2} \), \( \sech x = \frac{\sqrt{5}}{2} \), \( \coth x = \frac{\sqrt{5}}{2} \).

24 \( \sinh(\ln 5) = \frac{e^{\ln 5} - e^{-\ln 5}}{2} = \frac{5 - \frac{1}{5}}{2} = \frac{12}{5} \), \( \tanh(2\ln 4) = \frac{e^{2\ln 4} - e^{-2\ln 4}}{e^{2\ln 4} + e^{-2\ln 4}} = \frac{16 - \frac{1}{16}}{16 + \frac{1}{16}} = 255 \).

26 \( \int \cosh(x^2) \, dx = \frac{1}{2} \sinh(x^2) + C \). 28 \( \frac{1}{3} (\tanh x)^3 + C \).

30 \( \int \coth x \, dx = \int \frac{x}{\sinh x} \, dx = \ln(\sinh x) + C \). 32 \( \sinh x + \cosh x = e^x \) and \( \int e^{nx} \, dx = \frac{1}{n} e^{nx} + C \).

34 \( y = \tanh x \) is an odd function, with asymptote \( y = 1 \) as \( x \to -\infty \) and \( y = -1 \) as \( x \to +\infty \). The inflection point is \((0,0)\).

36 \( y = \sech x \) looks like a bell-shaped curve with \( y_{\text{max}} = 1 \) at \( x = 0 \). The \( x \) axis is the asymptote. But note that \( y \) decays like \( 2e^{-x} \) and not like \( e^{-x} \).

38 To define \( y = \cosh^{-1} x \) we require \( x \geq 1 \). Select the positive \( y \) (there are two \( y \)'s so strictly there is no inverse).

For large values, \( \cosh y \) is close to \( xe^y \), \( x \) is close to \( \ln \left( \frac{e^y}{2} \right) \), so \( \cosh^{-1} x \) to \( \frac{1}{2} \).

40 \( \frac{1}{2} \ln(\frac{1+x}{1-x}) \) approaches \( +\infty \) as \( x \to 1 \) and \( -\infty \) as \( x \to -1 \). The function is odd (so is the \( \tanh \) function).

The graph is an S curve rotated by \( 90^\circ \).

42 The quadratic equation for \( e^y \) has solution \( e^y = x \pm \sqrt{x^2 - 1} \). Choose the plus sign so \( y \to \infty \) as \( x \to -\infty \).

Then \( y = \ln(x + \sqrt{x^2 - 1}) \) is another form of \( y = \cosh^{-1} x \).

44 The \( x \) derivative of \( x = \sinh y \) is \( 1 = \cosh y \frac{dy}{dx} \). Then \( \frac{dy}{dx} = \frac{\cosh y}{\cosh^2 y} = \frac{1}{\sqrt{1 + x^2}} \) is the slope of \( \sinh^{-1} x \).

46 The \( x \) derivative of \( x = \sech y \) is \( 1 = -\sech y \tanh y \frac{dy}{dx} \). Then \( \frac{dy}{dx} = \frac{-1}{\sech y \tanh y} = \frac{-1}{x \sqrt{1-x^2}} \).

48 Set \( x = au \) and \( dx = du \) to reach \( \int \frac{x}{\sqrt{(1-u^2)}} \, du = \frac{1}{a} \tanh^{-1} u = \frac{1}{a} \tan^{-1} \frac{1-x}{a} + C \).

50 Not hyperbolic! Just \( f(x^2 + 1)^{-1/2} \, dx = (x^2 + 1)^{1/2} + C \).

52 Not hyperbolic! \( \int \frac{dx}{\sqrt{1-x^2}} = \sin^{-1} x + C \).

54 (a) \( \frac{dy}{dt} = (\sqrt{t})^2 \sech^2 \sqrt{t} \frac{dt}{\sqrt{t}} = g(1 - \tanh^2 \sqrt{t}) = g - v^2 \). (b) \( \int \frac{dy}{g - v^2} = \int dt \) gives (by Problem 48) \( \int \frac{1}{\sqrt{v}} \tanh^{-1} \sqrt{v} \, dt = \sqrt{v} \tanh^{-1} \sqrt{v} = t \) or \( \tan^{-1} \frac{\sqrt{v}}{\sqrt{t}} = \sqrt{t} \) or \( \sqrt{t} = \tanh \sqrt{t} \). (c) \( f(t) = \int \sqrt{t} \, \tanh \sqrt{t} \, dt = \int \sinh \sqrt{t} \, \sqrt{t} \, dt = \ln(\cosh \sqrt{t}) + C \).

56 Change to \( dx = \frac{dy}{\sqrt{y^2 + W^2}} = -\frac{dy}{2W - \frac{dy}{W}} \) and integrate: \( x = \ln(2 - W) - \ln W = \ln(\frac{2-W}{W}) \). Then \( \frac{2-W}{W} = e^x \) and \( W = \frac{2}{1+e^{-x}} \). (Note: The text suggests \( W = 2 \) but that is negative.

Writing \( \frac{2}{1+e^x} \) as \( e^{-x/2} \sech^{2x} \) is not simpler.)

58 \( \cos ix = \frac{1}{2}(e^{i(x)} + e^{-i(x)}) = \frac{1}{2}(e^{-x} + e^x) = \cos x \). Then \( \cos i = \cosh 1 = \frac{e^1 + e^{-1}}{2} \) (real!).

60 The derivative of \( e^{ix} = \cos x + i \sin x \) is \( i e^{ix} \) at \( x + i \sin x \) on the left side and \( \frac{d}{dx} \cos x + i \frac{d}{dx} \sin x \) on the right side. Comparing we again find \( \frac{d}{dx} (\sin x) = \cos x \) and \( \frac{d}{dx} (\cos x) = -\sin x \).