CHAPTER 14

Multiple Integrals

14.1 Double Integrals

This chapter shows how to integrate functions of two or more variables. First, a double integral is defined as the limit of sums. Second, we find a fast way to compute it. ***The key idea is to replace a double integral by two ordinary "single" integrals.***

The double integral $\iint f(x, y)dy\,dx$ starts with $\int f(x, y)dy$. For each fixed x we integrate with respect to y. The answer depends on x. Now integrate again, this time with respect to x. The limits of integration need care and attention! Frequently those limits on y and x are the hardest part.

Why bother with sums and limits in the first place? Two reasons. There has to be a definition and a computation to fall back on, when the single integrals are difficult or impossible. And also—this we emphasize—*multiple integrals represent more than area and volume.* Those words and the pictures that go with them are the easiest to understand. You can almost see the volume as a "sum of slices" or a "double sum of thin sticks." The true applications are mostly to other things, but the central idea is always the same: ***Add up small pieces and take limits.***

We begin with the area of R and the volume of V, by double integrals.

A LIMIT OF SUMS

The graph of $z = f(x, y)$ is a curved surface above the xy plane. At the point (x, y) in the plane, the height of the surface is z. (The surface is *above* the xy plane only when z is positive. Volumes below the plane come with minus signs, like areas below the x axis.) We begin by choosing a positive function—for example $z = 1 + x^2 + y^2$.

The base of our solid is a region R in the xy plane. That region will be chopped into small rectangles (sides Δx and Δy). When R itself is the rectangle $0 \leqslant x \leqslant 1$, $0 \leqslant y \leqslant 2$, the small pieces fit perfectly. For a triangle or a circle, the rectangles miss part of R. But they do fit in the limit, and any region with a piecewise smooth boundary will be acceptable.

Question What is the volume above R and below the graph of $z = f(x, y)$?

Answer It is a double integral—***the integral of*** $f(x, y)$ ***over*** R. To reach it we begin with a sum, as suggested by Figure 14.1.

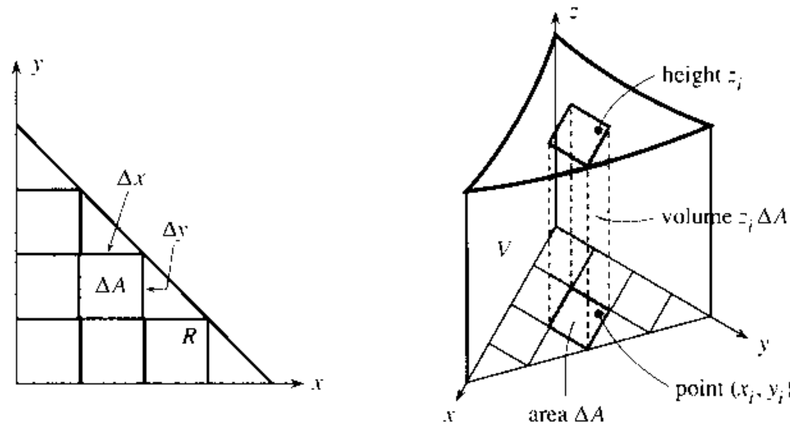

Fig. 14.1 Base R cut into small pieces ΔA. Solid V cut into thin sticks $\Delta V = z\,\Delta A$.

For single integrals, the interval $[a, b]$ is divided into short pieces of length Δx. For double integrals, R is divided into small rectangles of area $\Delta A = (\Delta x)(\Delta y)$. Above the ith rectangle is a "thin stick" with small volume. That volume is the base area ΔA times the height above it—except that this height $z = f(x, y)$ varies from point to point. Therefore we select a point (x_i, y_i) in the ith rectangle, and compute the volume from the height above that point:

$$\textit{volume of one stick} = f(x_i, y_i)\Delta A \qquad \textit{volume of all sticks} = \sum f(x_i, y_i)\Delta A.$$

This is the crucial step for any integral—to see it as a sum of small pieces.

Now take limits: $\Delta x \to 0$ and $\Delta y \to 0$. The height $z = f(x, y)$ is nearly constant over each rectangle. (We assume that f is a continuous function.) The sum approaches a limit, which depends only on the base R and the surface above it. The limit is the volume of the solid, and it is the ***double integral*** of $f(x, y)$ over R:

$$\iint_R f(x, y)\, dA = \lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} \sum f(x_i, y_i)\Delta A. \tag{1}$$

To repeat: The limit is the same for all choices of the rectangles and the points (x_i, y_i). The rectangles will not fit exactly into R, if that base area is curved. The heights are not exact, if the surface $z = f(x, y)$ is also curved. But the errors on the sides and top, where the pieces don't fit and the heights are wrong, approach zero. Those errors are the volume of the "icing" around the solid, which gets thinner as $\Delta x \to 0$ and $\Delta y \to 0$. A careful proof takes more space than we are willing to give. But the properties of the integral need and deserve attention:

1. Linearity: $\iint (f + g)\,dA = \iint f\,dA + \iint g\,dA$
2. Constant comes outside: $\iint cf(x, y)\,dA = c \iint f(x, y)\,dA$
3. R splits into S and T (not overlapping): $\iint_R f\,dA = \iint_S f\,dA + \iint_T f\,dA$.

In **1** the volume under $f + g$ has two parts. The "thin sticks" of height $f + g$ split into thin sticks under f and under g. In **2** the whole volume is stretched upward by c. In **3** the volumes are side by side. As with single integrals, these properties help in computations.

By writing dA, we allow shapes other than rectangles. Polar coordinates have an extra factor r in $dA = r\,dr\,d\theta$. By writing $dx\,dy$, we choose rectangular coordinates and prepare for the splitting that comes now.

SPLITTING A DOUBLE INTEGRAL INTO TWO SINGLE INTEGRALS

The double integral $\iint f(x, y)dy\,dx$ will now be reduced to single integrals in y and then x. (Or vice versa. Our first integral could equally well be $\int f(x, y)dx$.) Chapter 8 described the same idea for solids of revolution. First came the area of a slice, which is a single integral. Then came a second integral to add up the slices. For solids formed by revolving a curve, all slices are circular disks—now we expect other shapes.

Figure 14.2 shows a slice of area $A(x)$. It cuts through the solid at a fixed value of x. The cut starts at $y = c$ on one side of R, and ends at $y = d$ on the other side. This particular example goes from $y = 0$ to $y = 2$ (R is a rectangle). The area of a slice is the y integral of $f(x, y)$. *Remember that x is fixed and y goes from c to d*:

$$A(x) = \textbf{\textit{area of slice}} = \int_c^d f(x, y)dy \quad \text{(the answer is a function of } x\text{)}.$$

EXAMPLE 1 $$A = \int_{y=0}^{2} (1 + x^2 + y^2)dy = \left[y + x^2y + \frac{y^3}{3} \right]_{y=0}^{y=2} = 2 + 2x^2 + \frac{8}{3}.$$

This is the reverse of a partial derivative! The integral of x^2dy, with x constant, is x^2y. This "partial integral" is actually called an ***inner integral***. After substituting the limits $y = 2$ and $y = 0$ and subtracting, we have the area $A(x) = 2 + 2x^2 + \frac{8}{3}$. Now the ***outer integral*** adds slices to find the volume $\int A(x)\,dx$. The answer is a *number*:

$$\text{volume} = \int_{x=0}^{1} \left(2 + 2x^2 + \frac{8}{3}\right)dx = \left[2x + \frac{2x^3}{3} + \frac{8}{3}x \right]_0^1 = 2 + \frac{2}{3} + \frac{8}{3} = \frac{16}{3}.$$

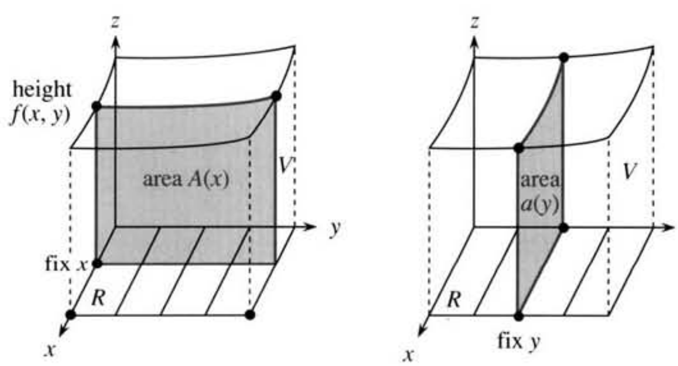

Fig. 14.2 A slice of V at a fixed x has area $A(x) = \int f(x, y)dy$.

To complete this example, check the volume when the x integral comes first:

$$\text{inner integral} = \int_{x=0}^{1} (1 + x^2 + y^2)dx = \left[x + \frac{1}{3}x^3 + y^2x \right]_{x=0}^{x=1} = \frac{4}{3} + y^2$$

$$\text{outer integral} = \int_{y=0}^{2} \left(\frac{4}{3} + y^2\right)dy = \left[\frac{4}{3}y + \frac{1}{3}y^3 \right]_{y=0}^{y=2} = \frac{8}{3} + \frac{8}{3} = \frac{16}{3}.$$

The fact that double integrals can be split into single integrals is *Fubini's Theorem*.

14A If $f(x, y)$ is continuous on the rectangle R, then

$$\iint_R f(x, y)dA = \int_a^b \left[\int_c^d f(x, y)dy \right] dx = \int_c^d \left[\int_a^b f(x, y)dx \right] dy. \qquad (2)$$

The inner integrals are the cross-sectional areas $A(x)$ and $a(y)$ of the slices. The outer integrals add up the volumes $A(x)dx$ and $a(y)dy$. Notice the reversing of limits.

Normally the brackets in (2) are omitted. When the y integral is first, dy is written inside dx. ***The limits on y are inside too.*** I strongly recommend that you compute the inner integral on one line and the outer integral on a *separate line*.

EXAMPLE 2 Find the volume below the plane $z = x - 2y$ and above the base triangle R.

The triangle R has sides on the x and y axes and the line $x + y = 1$. The strips in the y direction have varying lengths. (So do the strips in the x direction.) This is the main point of the example—the base is not a rectangle. The upper limit on the inner integral changes as x changes. ***The top of the triangle is at*** $y = 1 - x$.

Figure 14.3 shows the strips. The region should always be drawn (except for rectangles). Without a figure the limits are hard to find. A sketch of R makes it easy:

y goes from $c = 0$ to $d = 1 - x$. Then x goes from $a = 0$ to $b = 1$.

The inner integral has *variable limits* and the outer integral has *constant limits*:

$$\text{inner: } \int_{y=0}^{y=1-x} (x - 2y)dy = \left[xy - y^2\right]_{y=0}^{y=1-x} = x(1-x) - (1-x)^2 = -1 + 3x - 2x^2$$

$$\text{outer: } \int_{x=0}^{1} (-1 + 3x - 2x^2)dx = \left[-x + \frac{3}{2}x^2 - \frac{2}{3}x^3\right]_0^1 = -1 + \frac{3}{2} - \frac{2}{3} = -\frac{1}{6}.$$

The volume is negative. Most of the solid is below the xy plane. To check the answer $-\frac{1}{6}$, do the x integral first: *x goes from 0 to $1 - y$. Then y goes from 0 to 1.*

$$\text{inner: } \int_{x=0}^{1-y} (x - 2y)dx = \left[\frac{1}{2}x^2 - 2xy\right]_0^{1-y} = \frac{1}{2}(1-y)^2 - 2(1-y)y = \frac{1}{2} - 3y + \frac{5}{2}y^2$$

$$\text{outer: } \int_{y=0}^{1} \left(\frac{1}{2} - 3y + \frac{5}{2}y^2\right)dy = \left[\frac{1}{2}y - \frac{3}{2}y^2 + \frac{5}{6}y^3\right]_0^1 = \frac{1}{2} - \frac{3}{2} + \frac{5}{6} = -\frac{1}{6}.$$

Same answer, very probably right. The next example computes $\iint 1\, dx\, dy =$ area of R.

EXAMPLE 3 The area of R is $\displaystyle\int_{x=0}^{1}\int_{y=0}^{1-x} dy\, dx$ and also $\displaystyle\int_{y=0}^{1}\int_{x=0}^{1-y} dx\, dy$.

The first has vertical strips. The inner integral equals $1 - x$. Then the outer integral (of $1 - x$) has limits 0 and 1, and the area is $\frac{1}{2}$. It is like an indefinite integral inside a definite integral.

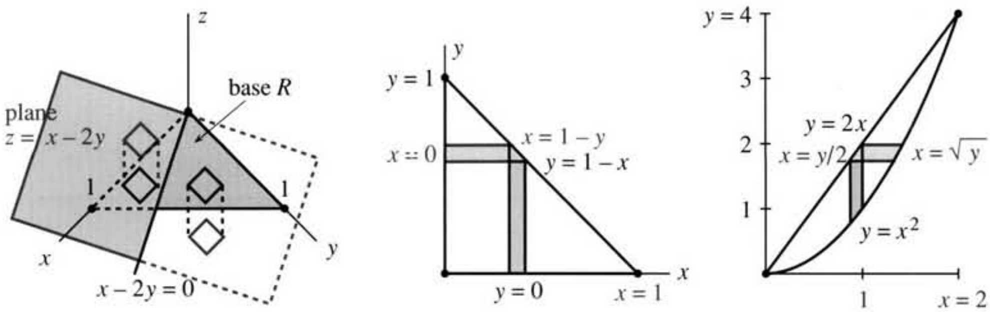

Fig. 14.3 Thin sticks above and below (Example 2). Reversed order (Examples 3 and 4).

EXAMPLE 4 Reverse the order of integration in $\int_{x=0}^{2}\int_{y=x^2}^{2x} x^3 dy\, dx$.

Solution Draw a figure! The inner integral goes from the parabola $y = x^2$ up to the straight line $y = 2x$. This gives vertical strips. The strips sit side by side between $x = 0$ and $x = 2$. They stop where $2x$ equals x^2, and the line meets the parabola.

The problem is to put the x integral first. It goes along horizontal strips. On each line y = constant, we need the *entry value* of x and the *exit value* of x. From the figure, x goes from $\frac{1}{2}y$ to $\sqrt{y}$. Those are the inner limits. Pay attention also to the outer limits, because they now apply to y. The region starts at $y = 0$ and ends at $y = 4$. *No change in the integrand* x^3—that is the height of the solid:

$$\int_{x=0}^{2}\int_{y=x^2}^{2x} x^3 dy\, dx \quad \textit{is reversed to} \quad \int_{y=0}^{4}\int_{x=\frac{1}{2}y}^{\sqrt{y}} x^3 dx\, dy. \tag{3}$$

EXAMPLE 5 Find the volume bounded by the planes $x = 0$, $y = 0$, $z = 0$, and $2x + y + z = 4$.

Solution The solid is a tetrahedron (four sides). It goes from $z = 0$ (the xy plane) up to the plane $2x + y + z = 4$. On that plane $z = 4 - 2x - y$. This is the height function $f(x, y)$ to be integrated.

Figure 14.4 shows the base R. To find its sides, set $z = 0$. The sides of R are the lines $x = 0$ and $y = 0$ and $2x + y = 4$. Taking vertical strips, dy is inner:

$$\text{inner: } \int_{y=0}^{4-2x} (4 - 2x - y)dy = \left[(4 - 2x)y - \frac{1}{2}y^2\right]_0^{4-2x} = \frac{1}{2}(4 - 2x)^2$$

$$\text{outer: } \int_{x=0}^{2} \frac{1}{2}(4 - 2x)^2 dx = \left[-\frac{(4 - 2x)^3}{2\cdot 3\cdot 2}\right]_0^2 = \frac{4^3}{2\cdot 3\cdot 2} = \frac{16}{3}.$$

Question What is the meaning of the inner integral $\frac{1}{2}(4 - 2x)^2$ $\left(\text{and also } \frac{16}{3}\right)$?

Answer The first is $A(x)$, the area of the slice. $\frac{16}{3}$ is the solid volume.

Question What if the inner integral $\int f(x, y)dy$ has limits that depend on y?
Answer It can't. Those limits must be wrong. Find them again.

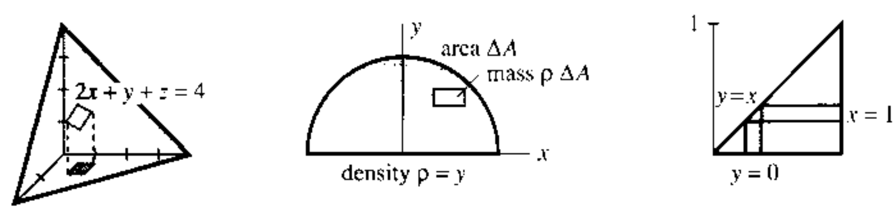

Fig. 14.4 Tetrahedron in Example 5, semicircle in Example 6, triangle in Example 7.

EXAMPLE 6 Find the mass in a semicircle $0 \leqslant y \leqslant \sqrt{1 - x^2}$ if the density is $\rho = y$.

This is a new application of double integrals. The total mass is a sum of small masses (ρ times ΔA) in rectangles of area ΔA. The rectangles don't fit perfectly inside the semicircle R, and the density is not constant in each rectangle—but those problems disappear in the limit. We are left with a double integral:

$$\textbf{\textit{total mass}}\ M = \iint_R \rho\, dA = \iint_R \rho(x, y)\, dx\, dy. \tag{4}$$

Set $\rho = y$. Figure 14.4 shows the limits on x and y (try both $dy\,dx$ and $dx\,dy$):

$$\text{mass } M = \int_{x=-1}^{1} \int_{y=0}^{\sqrt{1-x^2}} y\, dy\, dx \quad \text{and also} \quad M = \int_{y=0}^{1} \int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} y\, dx\, dy.$$

The first inner integral is $\frac{1}{2}y^2$. Substituting the limits gives $\frac{1}{2}(1-x^2)$. The outer integral of $\frac{1}{2}(1-x^2)$ yields the total mass $M = \frac{2}{3}$.

The second inner integral is xy. Substituting the limits on x gives _______. Then the outer integral is $-\frac{2}{3}(1-y^2)^{3/2}$. Substituting $y = 1$ and $y = 0$ yields $M =$ _______.

Remark This same calculation also produces the *moment* around the x axis, when the density is $\rho = 1$. The factor y is the distance to the x axis. ***The moment is*** $M_x = \iint y\, dA = \frac{2}{3}$. Dividing by the area of the semicircle (which is $\pi/2$) locates the centroid: $\bar{x} = 0$ by symmetry and

$$\bar{y} = \text{height of centroid} = \frac{\text{moment}}{\text{area}} = \frac{2/3}{\pi/2} = \frac{4}{3\pi}. \tag{5}$$

This is the "average height" of points inside the semicircle, found earlier in 8.5.

EXAMPLE 7 Integrate $\int_{y=0}^{y=1} \int_{x=y}^{x=1} \cos x^2 dx\, dy$ avoiding the impossible $\int \cos x^2\, dx$.

This is a famous example where reversing the order makes the calculation possible. The base R is the triangle in Figure 14.4 (note that x goes from y to 1). ***In the opposite order*** y ***goes from*** 0 ***to*** x. Then $\int \cos x^2 dy = x \cos x^2$ contains the factor x that we need:

$$\text{outer integral: } \int_0^1 x \cos x^2 dx = \left[\tfrac{1}{2} \sin x^2\right]_0^1 = \tfrac{1}{2} \sin 1.$$

14.1 EXERCISES

Read-through questions

The double integral $\iint_R f(x, y)dA$ gives the volume between R and __a__. The base is first cut into small __b__ of area ΔA. The volume above the ith piece is approximately __c__. The limit of the sum __d__ is the volume integral. Three properties of double integrals are __e__ (linearity) and __f__ and __g__.

If R is the rectangle $0 \leqslant x \leqslant 4$, $4 \leqslant y \leqslant 6$, the integral $\iint x\, dA$ can be computed two ways. One is $\iint x\, dy\, dx$, when the inner integral is __h__$]_4^6 =$ __i__. The outer integral gives __j__$]_0^4 =$ __k__. When the x integral comes first it equals $\int x\, dx =$ __l__$]_0^4 =$ __m__. Then the y integral equals __n__. This is the volume between __o__ (describe V).

The area of R is $\iint$ __p__ $dy\, dx$. When R is the triangle between $x = 0$, $y = 2x$, and $y = 1$, the inner limits on y are __q__. This is the length of a __r__ strip. The (outer) limits on x are __s__. The area is __t__. In the opposite order, the (inner) limits on x are __u__. Now the strip is __v__ and the outer integral is __w__. When the density is $\rho(x, y)$, the total mass in the region R is $\iint$ __x__. The moments are $M_y =$ __y__ and $M_x =$ __z__. The centroid has $\bar{x} = M_y/M$.

Compute the double integrals 1–4 by two integrations.

1 $\displaystyle\int_{y=0}^{1} \int_{x=0}^{2} x^2 dx\, dy$ and $\displaystyle\int_{y=0}^{1} \int_{x=0}^{2} y^2 dx\, dy$

2 $\displaystyle\int_{y=2}^{2e} \int_{x=1}^{e} 2xy\, dx\, dy$ and $\displaystyle\int_{y=2}^{2e} \int_{x=1}^{e} dx\, dy/xy$

3 $\displaystyle\int_0^{\pi/2} \int_0^{\pi/4} \sin(x + y)\, dx\, dy$ and $\displaystyle\int_1^2 \int_0^2 dy\, dx/(x + y)^2$

4 $\displaystyle\int_0^1 \int_1^2 ye^{xy} dx\, dy$ and $\displaystyle\int_{-1}^{1} \int_0^3 dy\, dx/\sqrt{3 + 2x + y}$

In 5–10, draw the region and compute the area.

5 $\int_{x=1}^{2}\int_{y=1}^{2x} dy\,dx$ 6 $\int_0^1\int_{x^3}^{x} dy\,dx$

7 $\int_0^{\infty}\int_{e^{-2x}}^{e^{-x}} dy\,dx$ 8 $\int_{-1}^{1}\int_{x^2-1}^{1-x^2} dy\,dx$

9 $\int_{-1}^{1}\int_{y^2}^{1} dx\,dy$ 10 $\int_{-1}^{1}\int_{x=y}^{|y|} dx\,dy$

In 11–16 reverse the order of integration (and find the new limits) in 5–10 respectively.

In 17–24 find the limits on $\iint dy\,dx$ and $\iint dx\,dy$. Draw R and compute its area.

17 R = triangle inside the lines $x=0$, $y=1$, $y=2x$.

18 R = triangle inside the lines $x=-1$, $y=0$, $x+y=0$.

19 R = triangle inside the lines $y=x$, $y=-x$, $y=3$.

20 R = triangle inside the lines $y=x$, $y=2x$, $y=4$.

21 R = triangle with vertices (0, 0), (4, 4), (4, 8).

22 R = triangle with vertices (0, 0), (−2, −1), (1, −2).

23 R = triangle with vertices (0, 0), (2, 0), (1, b). Here $b>0$.

*24 R = triangle with vertices (0, 0), (a, b), (c, d). The sides are $y=bx/a$, $y=dx/c$, and $y=b+(x-a)(d-b)/(c-a)$. Find $A=\iint dy\,dx$ when $0<a<c$, $0<d<b$.

25 Evaluate $\int_0^b\int_0^a \partial^2 f/\partial x\partial y\,dx\,dy$.

26 Evaluate $\int_0^b\int_0^a \partial f/\partial x\,dx\,dy$.

In 27–28, divide the unit square R into triangles S and T and verify $\iint_R f\,dA=\iint_S f\,dA+\iint_T f\,dA$.

27 $f(x, y)=2x-3y+1$ 28 $f(x, y)=xe^y-ye^x$

29 The area under $y=f(x)$ is a single integral from a to b or a double integral (*find the limits*):

$$\int_a^b f(x)\,dx=\iint 1\,dy\,dx.$$

30 Find the limits and the area under $y=1-x^2$:

$$\int(1-x^2)\,dx \text{ and } \iint 1\,dx\,dy \text{ (reversed from 29).}$$

31 A city inside the circle $x^2+y^2=100$ has population density $\rho(x, y)=10(100-x^2-y^2)$. Integrate to find its population.

32 Find the volume bounded by the planes $x=0$, $y=0$, $z=0$, and $ax+by+cz=1$.

In 33–34 the rectangle with corners (1, 1), (1, 3), (2, 1), (2, 3) has density $\rho(x, y)=x^2$. The moments are $M_y=\iint x\rho\,dA$ and $M_x=\iint y\rho\,dA$.

33 Find the mass. 34 Find the center of mass.

In 35–36 the region is a circular wedge of radius 1 between the lines $y=x$ and $y=-x$.

35 Find the area. 36 Find the centroid $(\bar{x}, \bar{y})$.

37 Write a program to compute $\int_0^1\int_0^1 f(x, y)dx\,dy$ by the midpoint rule (*midpoints of n^2 small squares*). Which $f(x, y)$ are integrated exactly by your program?

38 Apply the midpoint code to integrate x^2 and xy and y^2. The errors decrease like what power of $\Delta x=\Delta y=1/n$?

Use the program to compute the volume under $f(x, y)$ in 39–42. Check by integrating exactly or doubling n.

39 $f(x, y)=3x+4y+5$ 40 $f(x, y)=1/\sqrt{x^2+y^2}$

41 $f(x, y)=x^y$ 42 $f(x, y)=e^x\sin\pi y$

43 In which order is $\iint x^y dx\,dy=\iint x^y dy\,dx$ easier to integrate over the square $0\leqslant x\leqslant 1$, $0\leqslant y\leqslant 1$? By reversing order, integrate $(x-1)/\ln x$ from 0 to 1—its antiderivative is unknown.

44 Explain in your own words the definition of the double integral of $f(x, y)$ over the region R.

45 $\sum y_i\Delta A$ might not approach $\iint y\,dA$ if we only know that $\Delta A\to 0$. In the square $0\leqslant x, y\leqslant 1$, take rectangles of sides Δx and 1 (not Δx and Δy). If (x_i, y_i) is a point in the rectangle where $y_i=1$, then $\sum y_i\Delta A=$ ______. But $\iint y\,dA=$ ______.

14.2 Change to Better Coordinates

You don't go far with double integrals before wanting to ***change variables***. Many regions simply do not fit with the x and y axes. Two examples are in Figure 14.5, a tilted square and a ring. Those are excellent shapes—***in the right coordinates***.

We have to be able to answer basic questions like these:

Find the area $\iint dA$ and moment $\iint x\,dA$ and moment of inertia $\iint x^2\,dA$.

The problem is: *What is dA*? We are leaving the xy variables where $dA = dx\,dy$.

The reason for changing is this: The limits of integration in the y direction are miserable. I don't know them and I don't want to know them. For every x we would need the entry point P of the line $x = $ constant, and the exit point Q. The heights of P and Q are the limits on $\int dy$, the inner integral. The geometry of the square and ring are totally missed, if we stick rigidly to x and y.

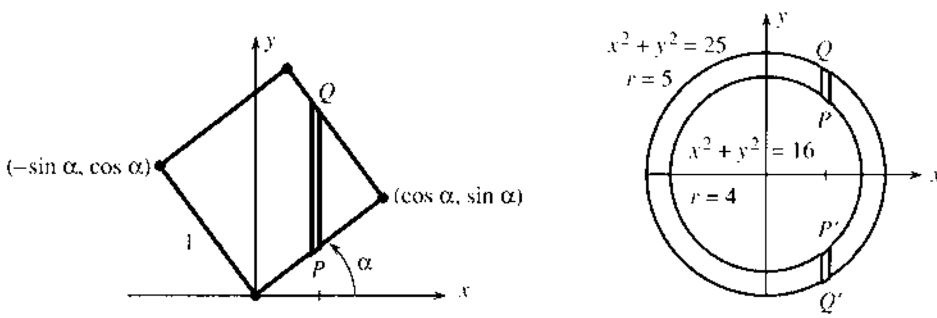

Fig. 14.5 Unit square turned through angle α. Ring with radii 4 and 5.

Which coordinates are better? Any sensible person agrees that the area of the tilted square is 1. "Just turn it and the area is obvious." But that sensible person may not know the moment or the center of gravity or the moment of inertia. So we actually have to do the turning.

The new coordinates u and v are in Figure 14.6a. The limits of integration on v are 0 and 1. So are the limits on u. *But when you change variables, you don't just change limits.* Two other changes come with new variables:

1. The small area $dA = dx\,dy$ becomes $dA =$ _______ $du\,dv$.
2. The integral of x becomes the integral of _______.

Substituting $u = \sqrt{x}$ in a single integral, we make the same changes. Limits $x = 0$ and $x = 4$ become $u = 0$ and $u = 2$. Since x is u^2, dx is $2u\,du$. The purpose of the change is to find an antiderivative. For double integrals, the usual purpose is to improve the limits—but we have to accept the whole package.

To turn the square, there are formulas connecting x and y to u and v. The geometry is clear—***rotate axes by*** α—but it has to be converted into algebra:

$$\begin{aligned} u &= x\cos\alpha + y\sin\alpha \\ v &= -x\sin\alpha + y\cos\alpha \end{aligned} \quad \text{and in reverse} \quad \begin{aligned} x &= u\cos\alpha - v\sin\alpha \\ y &= u\sin\alpha + v\cos\alpha. \end{aligned} \qquad (1)$$

Figure 14.6 shows the rotation. As points move, the whole square turns. A good way to remember equation (1) is to follow the corners as they become (1, 0) and (0, 1).

The change from $\iint x\,dA$ to $\iint$ _______ $du\,dv$ is partly decided by equation (1). It gives x as a function of u and v. We also need dA. For a pure rotation the first guess is correct: ***The area $dx\,dy$ equals the area $du\,dv$. For most changes of variable this is false.*** The general formula for dA comes after the examples.

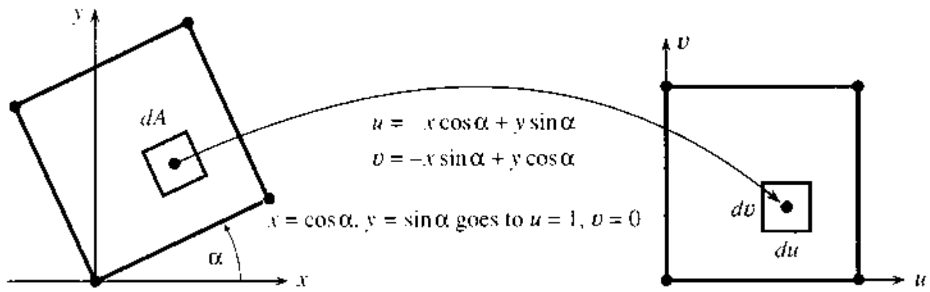

Fig. 14.6 Change of coordinates—axes turned by α. For rotation dA is $du\,dv$.

EXAMPLE 1 Find $\iint dA$ and $\iint x\,dA$ and $\bar{x}$ and also $\iint x^2\,dA$ for the tilted square.

Solution The area of the square is $\int_0^1\int_0^1 du\,dv = 1$. Notice the good limits. Then

$$\iint x\,dA = \int_0^1\int_0^1 (u\cos\alpha - v\sin\alpha)du\,dv = \tfrac{1}{2}\cos\alpha - \tfrac{1}{2}\sin\alpha. \qquad (2)$$

This is the *moment around the y axis*. The factors $\frac{1}{2}$ come from $\frac{1}{2}u^2$ and $\frac{1}{2}v^2$. The x coordinate of the center of gravity is

$$\bar{x} = \iint x\,dA \Big/ \iint dA = (\tfrac{1}{2}\cos\alpha - \tfrac{1}{2}\sin\alpha)/1.$$

Similarly the integral of y leads to $\bar{y}$. The answer is no mystery—the point $(\bar{x}, \bar{y})$ is at the center of the square! Substituting $x = u\cos\alpha - v\sin\alpha$ made $x\,dA$ look worse, but the limits 0 and 1 are much better.

The moment of inertia I_y around the y axis is also simplified:

$$\iint x^2\,dA = \int_0^1\int_0^1 (u\cos\alpha - v\sin\alpha)^2 du\,dv = \frac{\cos^2\alpha}{3} - \frac{\cos\alpha\sin\alpha}{2} + \frac{\sin^2\alpha}{3}. \qquad (3)$$

You know this next fact but I will write it anyway: *The answers don't contain u or v.* Those are dummy variables like x and y. The answers do contain α, because the square has turned. (The area is fixed at 1.) The moment of inertia $I_x = \iint y^2\,dA$ is the same as equation (3) but with all plus signs.

Question The sum $I_x + I_y$ simplifies to $\frac{2}{3}$ (a constant). Why no dependence on α?
Answer $I_x + I_y$ equals I_0. This moment of inertia around (0, 0) is unchanged by rotation. We are turning the square around one of its corners.

CHANGE TO POLAR COORDINATES

The next change is to r and θ. A small area becomes $dA = r\,dr\,d\theta$ (definitely not $dr\,d\theta$). Area always comes from multiplying two lengths, and $d\theta$ is not a length. Figure 14.7 shows the crucial region—a "polar rectangle" cut out by rays and circles. Its area ΔA is found in two ways, both leading to $r\,dr\,d\theta$:

(***Approximate***) The straight sides have length Δr. The circular arcs are *close to* $r\Delta\theta$. The angles are 90°. So ΔA is close to $(\Delta r)(r\Delta\theta)$.

(***Exact***) A wedge has area $\frac{1}{2}r^2\Delta\theta$. The difference between wedges is ΔA:

$$\Delta A = \frac{1}{2}\left(r + \frac{\Delta r}{2}\right)^2 \Delta\theta - \frac{1}{2}\left(r - \frac{\Delta r}{2}\right)^2 \Delta\theta = r\,\Delta r\,\Delta\theta.$$

The exact method places r dead center (see figure). The approximation says: Forget the change in $r\Delta\theta$ as you move outward. Keep only the first-order terms.

A third method is coming, which requires no picture and no geometry. Calculus always has a third method! The change of variables $x = r\cos\theta$, $y = r\sin\theta$ will go into a general formula for dA, and out will come the area $r\,dr\,d\theta$.

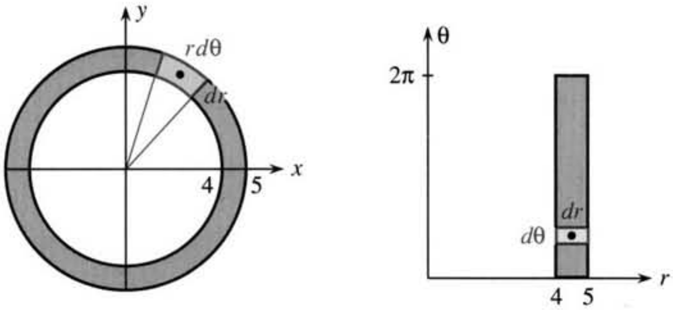

Fig. 14.7 Ring and polar rectangle in xy and $r\theta$, with stretching factor $r = 4.5$.

EXAMPLE 2 Find the area and center of gravity of the ring. Also find $\iint x^2 dA$.

Solution The limits on r are 4 and 5. The limits on θ are 0 and 2π. *Polar coordinates are perfect for a ring*. Compared with limits like $x = \sqrt{25 - y^2}$, the change to $r\,dr\,d\theta$ is a small price to pay:

$$\text{area} = \int_0^{2\pi}\int_4^5 r\,dr\,d\theta = 2\pi\left[\tfrac{1}{2}r^2\right]_4^5 = \pi 5^2 - \pi 4^2 = 9\pi.$$

The θ integral is 2π (full circle). Actually the ring is a giant polar rectangle. We could have used the exact formula $r\,\Delta r\,\Delta\theta$, with $\Delta\theta = 2\pi$ and $\Delta r = 5 - 4$. When the radius r is centered at 4.5, the product $r\,\Delta r\,\Delta\theta$ is $(4.5)(1)(2\pi) = 9\pi$ as above.

Since the ring is symmetric around (0, 0), the integral of $x\,dA$ must be *zero*:

$$\iint_R x\,dA = \int_0^{2\pi}\int_4^5 (r\cos\theta)r\,dr\,d\theta = \left[\tfrac{1}{3}r^3\right]_4^5 \left[\sin\theta\right]_0^{2\pi} = 0.$$

Notice $r\cos\theta$ from x—the other r is from dA. The moment of inertia is

$$\iint_R x^2 dA = \int_0^{2\pi}\int_4^5 r^2\cos^2\theta\; r\,dr\,d\theta = \left[\tfrac{1}{4}r^4\right]_4^5 \int_0^{2\pi}\cos^2\theta\,d\theta = \tfrac{1}{4}(5^4 - 4^4)\pi.$$

This θ integral is π not 2π, because the average of $\cos^2\theta$ is $\frac{1}{2}$ not 1.

For reference here are the moments of inertia when the density is $\rho(x, y)$:

$$I_y = \iint x^2\rho\,dA \qquad I_x = \iint y^2\rho\,dA \qquad I_0 = \iint r^2\rho\,dA = \textbf{polar moment} = I_x + I_y. \tag{4}$$

EXAMPLE 3 Find masses and moments for semicircular plates: $\rho = 1$ and $\rho = 1 - r$.

Solution The semicircles in Figure 14.8 have $r = 1$. The angle goes from 0 to π (the upper half-circle). Polar coordinates are best. ***The mass is the integral of the density*** ρ:

$$M = \int_0^{\pi}\int_0^1 r\,dr\,d\theta = (\tfrac{1}{2})(\pi) \qquad \text{and} \qquad M = \int_0^{\pi}\int_0^1 (1 - r)r\,dr\,d\theta = (\tfrac{1}{6})(\pi).$$

The first mass $\pi/2$ equals the area (because $\rho = 1$). The second mass $\pi/6$ is smaller (because $\rho < 1$). Integrating $\rho = 1$ is the same as finding a volume when the height is $z = 1$ (part of a cylinder). Integrating $\rho = 1 - r$ is the same as finding a volume when the height is $z = 1 - r$ (part of a cone). Volumes of cones have the extra factor $\frac{1}{3}$.

The center of gravity involves the moment $M_x = \iint y\rho \, dA$. ***The distance from the x axis is y, the mass of a small piece is ρ dA, integrate to add mass times distance.*** Polar coordinates are still best, with $y = r \sin\theta$. Again $\rho = 1$ and $\rho = 1 - r$:

$$\iint y \, dA = \int_0^\pi \int_0^1 r \sin\theta \, r \, dr \, d\theta = \tfrac{2}{3} \qquad \iint y(1-r) \, dA = \int_0^\pi \int_0^1 r \sin\theta (1-r) r \, dr \, d\theta = \tfrac{1}{6}.$$

The height of the center of gravity is $\bar{y} = M_x/M =$ ***moment divided by mass***:

$$\bar{y} = \frac{2/3}{\pi/2} = \frac{4}{3\pi} \quad \text{when } \rho = 1 \qquad \bar{y} = \frac{1/6}{\pi/6} = \frac{1}{\pi} \quad \text{when } \rho = 1 - r.$$

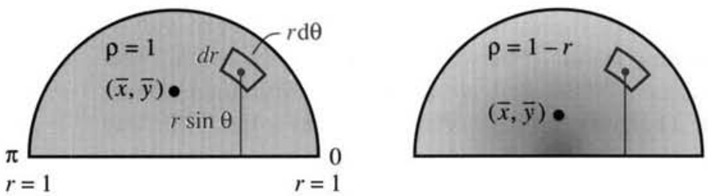

Fig. 14.8 Semicircles with density piled above them.

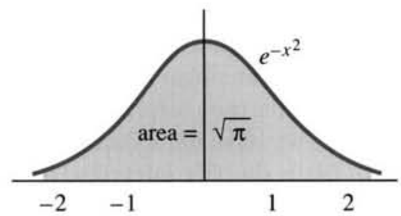

Fig. 14.9 Bell-shaped curve.

Question Compare $\bar{y}$ for $\rho = 1$ and $\rho =$ other positive constants and $\rho = 1 - r$.
Answer Any constant ρ gives $\bar{y} = 4/3\pi$. Since $1 - r$ is dense at $r = 0$, $\bar{y}$ drops to $1/\pi$.

Question How is $\bar{y} = 4/3\pi$ related to the "average" of y in the semicircle?
Answer They are identical. This is the point of $\bar{y}$. Divide the integral by the area:

$$\textit{The average value of a function is } \iint f(x, y) dA \Big/ \iint dA. \tag{5}$$

The integral of f is divided by the integral of 1 (the area). In one dimension $\int_a^b v(x) \, dx$ was divided by $\int_a^b 1 \, dx$ (the length $b - a$). That gave the average value of $v(x)$ in Section 5.6. Equation (5) is the same idea for $f(x, y)$.

EXAMPLE 4 Compute $A = \int_{-\infty}^{\infty} e^{-x^2} dx = \sqrt{\pi}$ from $A^2 = \int_{-\infty}^{\infty} e^{-x^2} dx \int_{-\infty}^{\infty} e^{-y^2} dy = \pi$.

A is the area under a "bell-shaped curve"—see Figure 14.9. This is the most important definite integral in the study of probability. It is difficult because a factor $2x$ is not present. Integrating $2xe^{-x^2}$ gives $-e^{-x^2}$, but integrating e^{-x^2} is impossible—except approximately by a computer. How can we hope to show that A is exactly $\sqrt{\pi}$?

The trick is to go from an area integral A to a volume integral A^2. This is unusual (and hard to like), but the end justifies the means:

$$A^2 = \int_{x=-\infty}^{\infty} \int_{y=-\infty}^{\infty} e^{-x^2} e^{-y^2} dy \, dx = \int_{\theta=0}^{2\pi} \int_{r=0}^{\infty} e^{-r^2} r \, dr \, d\theta. \tag{6}$$

The double integrals cover the whole plane. The r^2 comes from $x^2 + y^2$, and the key factor r appears in polar coordinates. It is now possible to substitute $u = r^2$. The r integral is $\frac{1}{2}\int_0^\infty e^{-u} du = \frac{1}{2}$. The θ integral is 2π. The double integral is $(\frac{1}{2})(2\pi)$. Therefore $A^2 = \pi$ and the single integral is $A = \sqrt{\pi}$.

EXAMPLE 5 Apply Example 4 to the "***normal distribution***" $p(x) = e^{-x^2/2}/\sqrt{2\pi}$.

Section 8.4 discussed probability. It emphasized the importance of this particular $p(x)$. At that time we could not verify that $\int p(x)\,dx = 1$. Now we can:

$$x = \sqrt{2}y \quad \text{yields} \quad \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-x^2/2}dx = \frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty} e^{-y^2}dy = 1. \tag{7}$$

Question Why include the 2's in $p(x)$? The integral of $e^{-x^2}/\sqrt{\pi}$ also equals 1.

Answer With the 2's the "***variance***" is $\int x^2 p(x)\,dx = 1$. This is a convenient number.

CHANGE TO OTHER COORDINATES

A third method was promised, to find $r\,dr\,d\theta$ without a picture and without geometry. The method works directly from $x = r\cos\theta$ and $y = r\sin\theta$. It also finds the 1 in $du\,dv$, after a rotation of axes. Most important, this new method finds the factor J in the area $dA = J\,du\,dv$, for any change of variables. The change is from xy to uv.

For single integrals, the "***stretching factor***" J between the original dx and the new du is (not surprisingly) the ratio dx/du. Where we have dx, we write $(dx/du)du$. Where we have $(du/dx)dx$, we write du. That was the idea of substitutions—the main way to simplify integrals.

For double integrals the stretching factor appears in the area: $dx\,dy$ becomes $|J|\,du\,dv$. ***The old and new variables are related by*** $x = x(u, v)$ ***and*** $y = y(u, v)$. The point with coordinates u and v comes from the point with coordinates x and y. A whole region S, full of points in the uv plane, comes from the region R full of corresponding points in the xy plane. A small piece with area $|J|\,du\,dv$ comes from a small piece with area $dx\,dy$. *The formula for J is a two-dimensional version of dx/du.*

14B The stretching factor for area is the 2 by 2 ***Jacobian determinant*** $J(u, v)$:

$$J = \begin{vmatrix} \partial x/\partial u & \partial x/\partial v \\ \partial y/\partial u & \partial y/\partial v \end{vmatrix} = \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u}. \tag{8}$$

An integral over R in the xy plane becomes an integral over S in the uv plane:

$$\iint_R f(x, y)\,dx\,dy = \iint_S f(x(u, v), y(u, v))\,|J|\,du\,dv. \tag{9}$$

The determinant J is often written $\partial(x, y)/\partial(u, v)$, as a reminder that this stretching factor is like dx/du. *We require* $J \neq 0$. That keeps the stretching and shrinking under control.

You naturally ask: Why take the absolute value $|J|$ in equation (9)? Good question—it wasn't done for single integrals. The reason is in the limits of integration. The single integral $\int_0^1 dx$ is $\int_0^{-1}(-du)$ after changing x to $-u$. *We keep the minus sign and allow single integrals to run backward.* Double integrals could too, but normally they go left to right and down to up. We use the absolute value $|J|$ and run forward.

EXAMPLE 6 Polar coordinates have $x = u\cos v = r\cos\theta$ and $y = u\sin v = r\sin\theta$.

$$\text{With no geometry:} \qquad J = \begin{vmatrix} \partial x/\partial r & \partial x/\partial \theta \\ \partial y/\partial r & \partial y/\partial \theta \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r. \tag{10}$$

EXAMPLE 7 Find J for the *linear change* to $x = au + bv$ and $y = cu + dv$.

Ordinary determinant: $$J = \begin{vmatrix} \partial x/\partial u & \partial x/\partial v \\ \partial y/\partial u & \partial y/\partial v \end{vmatrix} = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc. \qquad (11)$$

Why make this simple change, in which a, b, c, d are all constant? It straightens parallelograms into squares (and rotates those squares). Figure 14.10 is typical.

Common sense indicated $J = 1$ for pure rotation—no change in area. Now $J = 1$ comes from equations (1) and (11), because $ad - bc$ is $\cos^2\alpha + \sin^2\alpha$.

In practice, xy rectangles generally go into uv rectangles. The sides can be curved (as in polar rectangles) but the angles are often 90°. The change is "*orthogonal*." The next example has angles that are not 90°, and J still gives the answer.

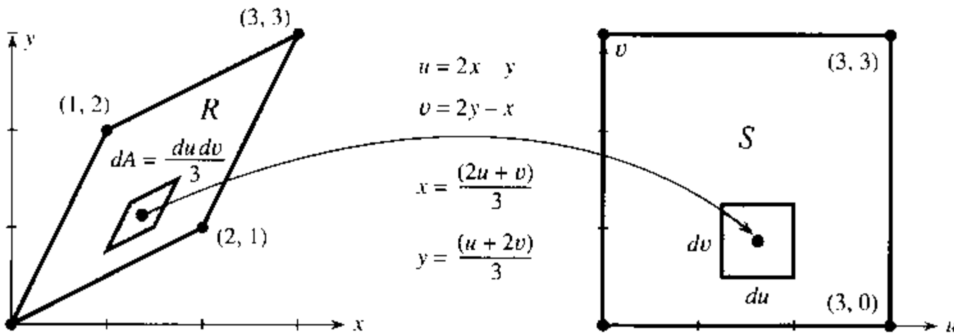

Fig. 14.10 Change from xy to uv has $J = \frac{1}{3}$.

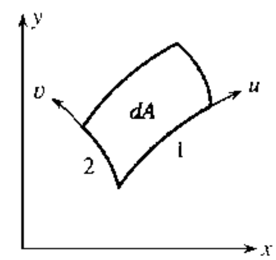

Fig. 14.11 Curved areas are also $dA = |J|du\,dv$.

EXAMPLE 8 Find the area of R in Figure 14.10. Also compute $\iint_R e^x dx\,dy$.

Solution The figure shows $x = \frac{2}{3}u + \frac{1}{3}v$ and $y = \frac{1}{3}u + \frac{2}{3}v$. The determinant is

$$J = \begin{vmatrix} \partial x/\partial u & \partial x/\partial v \\ \partial y/\partial u & \partial y/\partial v \end{vmatrix} = \begin{vmatrix} 2/3 & 1/3 \\ 1/3 & 2/3 \end{vmatrix} = \frac{4}{9} - \frac{1}{9} = \frac{1}{3}.$$

The area of the xy parallelogram becomes an integral over the uv square:

$$\iint_R dx\,dy = \iint_S |J|du\,dv = \int_0^3\int_0^3 \tfrac{1}{3}\,du\,dv = \tfrac{1}{3}\cdot 3\cdot 3 = 3.$$

The square has area 9, the parallelogram has area 3. I don't know if $J = \frac{1}{3}$ is a stretching factor or a shrinking factor. The other integral $\iint e^x dx\,dy$ is

$$\int_0^3\int_0^3 e^{2u/3 - v/3}\,\frac{1}{3}\,du\,dv = \left[\frac{3}{2}e^{2u/3}\right]_0^3 \left[3e^{v/3}\right]_0^3 \frac{1}{3} = \frac{3}{2}(e^2 - 1)(e - 1).$$

Main point: The change to u and v makes the limits easy (just 0 and 3).

Why is the stretching factor J a determinant? With straight sides, this goes back to Section 11.3 on vectors. *The area of a parallelogram is a determinant*. Here the sides are curved, but that only produces $(du)^2$ and $(dv)^2$, which we ignore.

A change du gives one side of Figure 14.11—it is $(\partial x/\partial u\,\mathbf{i} + \partial y/\partial u\,\mathbf{j})du$. Side 2 is $(\partial x/\partial v\,\mathbf{i} + \partial y/\partial v\,\mathbf{j})dv$. The curving comes from second derivatives. The area (the cross product of the sides) is $|J|du\,dv$.

Final remark I can't resist looking at the change in the reverse direction. Now the rectangle is in xy and the parallelogram is in uv. In all formulas, exchange x for u and y for v:

$$\textbf{\textit{new}}\ J=\begin{vmatrix}\partial u/\partial x & \partial u/\partial y\\ \partial v/\partial x & \partial v/\partial y\end{vmatrix}=\frac{\partial(u,v)}{\partial(x,y)}=\frac{1}{\textbf{\textit{old}}\ J}. \tag{12}$$

This is exactly like $du/dx=1/(dx/du)$. It is the derivative of the inverse function. The product of slopes is 1—stretch out, shrink back. From xy to uv we have 2 by 2 matrices, and the identity matrix I takes the place of 1:

$$\frac{dx}{du}\frac{du}{dx}=1 \quad\text{becomes}\quad \begin{bmatrix}\partial x/\partial u & \partial x/\partial v\\ \partial y/\partial u & \partial y/\partial v\end{bmatrix}\begin{bmatrix}\partial u/\partial x & \partial u/\partial y\\ \partial v/\partial x & \partial v/\partial y\end{bmatrix}=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}. \tag{13}$$

The first row times the first column is $(\partial x/\partial u)(\partial u/\partial x)+(\partial x/\partial v)(\partial v/\partial x)=\partial x/\partial x=1$. The first row times the second column is $(\partial x/\partial u)(\partial u/\partial y)+(\partial x/\partial v)(\partial v/\partial y)=\partial x/\partial y=0$. ***The matrices are inverses of each other.*** The determinants of a matrix and its inverse obey our rule: old J times new $J=1$. Those J's cannot be zero, just as dx/du and du/dx were not zero. (Inverse functions increase steadily or decrease steadily.)

In two dimensions, an area $dx\,dy$ goes to $J\,du\,dv$ and comes back to $dx\,dy$.

14.2 EXERCISES

Read-through questions

We change variables to improve the __a__ of integration. The disk $x^2+y^2\leqslant 9$ becomes the rectangle $0\leqslant r\leqslant$ __b__, $0\leqslant\theta\leqslant$ __c__. The inner limits on $\iint dy\,dx$ are $y=\pm$ __d__. In polar coordinates this area integral becomes $\iint$ __e__ $=$ __f__.

A polar rectangle has sides dr and __g__. Two sides are not __h__ but the angles are still __i__. The area between the circles $r=1$ and $r=3$ and the rays $\theta=0$ and $\theta=\pi/4$ is __j__. The integral $\iint x\,dy\,dx$ changes to $\iint$ __k__. This is the __l__ around the __m__ axis. Then $\bar{x}$ is the ratio __n__. This is the x coordinate of the __o__, and it is the __p__ value of x.

In a rotation through α, the point that reaches (u,v) starts at $x=u\cos\alpha-v\sin\alpha$, $y=$ __q__. A rectangle in the uv plane comes from a __r__ in xy. The areas are __s__ so the stretching factor is $J=$ __t__. This is the determinant of the matrix __u__ containing $\cos\alpha$ and $\sin\alpha$. The moment of inertia $\iint x^2dx\,dy$ changes to $\iint$ __v__ $du\,dv$.

For single integrals dx changes to __w__ du. For double integrals $dx\,dy$ changes to $J\,du\,dv$ with $J=$ __x__. The stretching factor J is the determinant of the 2 by 2 matrix __y__. The functions $x(u,v)$ and $y(u,v)$ connect an xy region R to a uv region S, and $\iint_R dx\,dy=\iint_S$ __z__ $=$ area of __A__. For polar coordinates $x=$ __B__, $y=$ __C__. For $x=u$, $y=u+4v$ the 2 by 2 determinant is $J=$ __D__. A square in the uv plane comes from a __E__ in xy. In the opposite direction the change has $u=x$ and $v=\frac{1}{4}(y-x)$ and a new $J=$ __F__. This J is constant because this change of variables is __G__.

In 1–12 R is a pie-shaped wedge: $0\leqslant r\leqslant 1$ and $\pi/4\leqslant\theta\leqslant 3\pi/4$.

1 What is the area of R? Check by integration in polar coordinates.

2 Find limits on $\iint dy\,dx$ to yield the area of R, and integrate. *Extra credit*: Find limits on $\iint dx\,dy$.

3 Equation (1) with $\alpha=\pi/4$ rotates R into the uv region $S=$ ________. Find limits on $\iint du\,dv$.

4 Compute the centroid height $\bar{y}$ of R by changing $\iint y\,dx\,dy$ to polar coordinates. Divide by the area of R.

5 The region R has $\bar{x}=0$ because ________. After rotation through $\alpha=\pi/4$, the centroid $(\bar{x},\bar{y})$ of R becomes the centroid ________ of S.

6 Find the centroid of any wedge $0\leqslant r\leqslant a$, $0\leqslant\theta\leqslant b$.

7 Suppose R^* is the wedge R moved up so that the sharp point is at $x=0$, $y=1$.

(a) Find limits on $\iint dy\,dx$ to integrate over R^*.

(b) With $x^*=x$ and $y^*=y-1$, the xy region R^* corresponds to what region in the x^*y^* plane?

(c) After that change $dx\,dy$ equals ________ dx^*dy^*.

8 Find limits on $\iint r\,dr\,d\theta$ to integrate over R^* in Problem 7.

9 The right coordinates for R^* are r^* and θ^*, with $x=r^*\cos\theta^*$ and $y=r^*\sin\theta^*+1$.

(a) Show that $J=r^*$ so $dA=r^*dr^*d\theta^*$.

(b) Find limits on $\iint r^*dr^*d\theta^*$ to integrate over R^*.

10 If the centroid of R is $(0, \bar{y})$, the centroid of R^* is ________. The centroid of the circle with radius 3 and center $(1, 2)$ is ________. The centroid of the upper half of that circle is ________.

11 The moments of inertia I_x, I_y, I_0 of the original wedge R are ________.

12 The moments of inertia I_x, I_y, I_0 of the shifted wedge R^* are ________.

Problems 13–16 change four-sided regions to squares.

13 R has straight sides $y = 2x, x = 1, y = 1 + 2x, x = 0$. Locate its four corners and draw R. Find its area by geometry.

14 Choose a, b, c, d so that the change $x = au + bv$, $y = cu + dv$ takes the previous R into S, the unit square $0 \leqslant u \leqslant 1$, $0 \leqslant v \leqslant 1$. From the stretching factor $J = ad - bc$ find the area of R.

15 The region R has straight sides $x = 0$, $x = 1$, $y = 0$, $y = 2x + 3$. Choose a, b, c so that $x = u$ and $y = au + bv + cuv$ change R to the unit square S.

16 A nonlinear term uv was needed in Problem 15. Which regions R could change to the square S with a linear $x = au + bv$, $y = cu + dv$?

Draw the xy region R that corresponds in 17–22 to the uv square S with corners $(0, 0)$, $(1, 0)$, $(0, 1)$, $(1, 1)$. Locate the corners of R and then its sides (like a jigsaw puzzle).

17 $x = 2u + v$, $y = u + 2v$

18 $x = 3u + 2v$, $y = u + v$

19 $x = e^{2u+v}$, $y = e^{u+2v}$

20 $x = uv$, $y = v^2 - u^2$

21 $x = u$, $y = v(1 + u^2)$

22 $x = u \cos v$, $y = u \sin v$ (only three corners)

23 In Problems 17 and 19, compute J from equation (8). Then find the area of R from $\iint_S |J|\, du\, dv$.

24 In 18 and 20, find $J = \partial(x, y)/\partial(u, v)$ and the area of R.

25 If R lies between $x = 0$ and $x = 1$ under the graph of $y = f(x) > 0$, then $x = u$, $y = vf(u)$ takes R to the unit square S. Locate the corners of R and the point corresponding to $u = \frac{1}{2}$, $v = 1$. Compute J to prove what we know:

$$\text{area of } R = \int_0^1 f(x)dx = \int_0^1 \int_0^1 J\, du\, dv.$$

26 From $r = \sqrt{x^2 + y^2}$ and $\theta = \tan^{-1}(y/x)$, compute $\partial r/\partial x$, $\partial r/\partial y$, $\partial \theta/\partial x$, $\partial \theta/\partial y$, and the determinant $J = \partial(r, \theta)/\partial(x, y)$. How is this J related to the factor $r = \partial(x, y)/\partial(r, \theta)$ that enters $r\, dr\, d\theta$?

27 Example 4 integrated e^{-x^2} from 0 to ∞ (answer $\sqrt{\pi}$). Also $B = \int_0^1 e^{-x^2} dx$ leads to $B^2 = \int_0^1 e^{-x^2} dx \int_0^1 e^{-y^2} dy$. Change this double integral over the unit square to r and θ— and find the limits on r that make exact integration impossible.

28 Integrate by parts to prove that the standard normal distribution $p(x) = e^{-x^2/2}/\sqrt{2\pi}$ has $\sigma^2 = \int_{-\infty}^{\infty} x^2 p(x)dx = 1$.

29 Find the average distance from a point on a circle to the points inside. Suggestion: Let $(0, 0)$ be the point and let $0 \leqslant r \leqslant 2a \cos \theta$, $0 \leqslant \theta \leqslant \pi$ be the circle (radius a). The distance is r, so the average distance is $\bar{r} = \iint$ ________ $/\iint$ ________.

30 Draw the region R: $0 \leqslant x \leqslant 1$, $0 \leqslant y < \infty$ and describe it with polar coordinates (limits on r and θ). Integrate $\iint_R (x^2 + y^2)^{-3/2} dx\, dy$ in polar coordinates.

31 Using polar coordinates, find the volume under $z = x^2 + y^2$ above the unit disk $x^2 + y^2 \leqslant 1$.

32 The end of Example 1 stated the moment of inertia $\iint y^2 dA$. Check that integration.

33 In the square $-1 \leqslant x \leqslant 2$, $-2 \leqslant y \leqslant 1$, where could you distribute a unit mass (with $\iint \rho\, dxdy = 1$) to maximize

(a) $\iint x^2 \rho\, dA$ (b) $\iint y^2 \rho\, dA$ (c) $\iint r^2 \rho\, dA$?

34 *True or false*, with a reason:

(a) If the uv region S corresponds to the xy region R, then area of S = area of R.

(b) $\iint x\, dA \leqslant \iint x^2 dA$

(c) The average value of $f(x, y)$ is $\iint f(x, y) dA$

(d) $\int_{-\infty}^{\infty} xe^{-x^2} dx = 0$

(e) A polar rectangle has the same area as a straight-sided region with the same corners.

35 Find the mass of the tilted square in Example 1 if the density is $\rho = xy$.

36 Find the mass of the ring in Example 2 if the density is $\rho = x^2 + y^2$. This is the same as which moment of inertia with which density?

37 Find the polar moment of inertia I_0 of the ring in Example 2 if the density is $\rho = x^2 + y^2$.

38 Give the following statement an appropriate name: $\iint_R f(x, y) dA = f(P)$ times (area of R), where P is a point in R. Which point P makes this correct for $f = x$ and $f = y$?

39 Find the xy coordinates of the top point in Figure 14.6a and check that it goes to $(u, v) = (1, 1)$.