

## Table of Integrals

$$\begin{aligned}
 1 \int u^n dx &= \frac{u^{n+1}}{a(n+1)} \quad \text{except for } \int \frac{dx}{u} = \frac{\ln|u|}{a} \quad \text{All the integrals 1 - 17 involve } u = ax + b \\
 2 \int xu^n dx &= \frac{u^{n+2}}{a^2(n+2)} - \frac{bu^{n+1}}{a^2(n+1)} \quad \text{except for } \int \frac{x dx}{u} = \frac{x}{a} - \frac{b \ln|u|}{a^2} \quad \text{and } \int \frac{x dx}{u^2} = \frac{b}{a^2 u} + \frac{\ln|u|}{a^2} \\
 3 \int \frac{x^2 dx}{u} &= \frac{1}{a^3} \left( \frac{u^2}{2} - 2bu + b^2 \ln|u| \right) \quad 4 \int \frac{x^2 dx}{u^2} = \frac{1}{a^3} \left( u - 2b \ln|u| - \frac{b^2}{u} \right) \quad 5 \int \frac{x^2 dx}{u^3} = \frac{1}{a^3} \left( \ln|u| + \frac{2b}{u} - \frac{b^2}{2u^2} \right) \\
 6 \int \frac{dx}{xu} &= \frac{1}{b} \ln \left| \frac{x}{u} \right| \quad 7 \int \frac{dx}{x^2 u} = -\frac{1}{bx} + \frac{a}{b^2} \ln \left| \frac{u}{x} \right| \quad 8 \int \frac{dx}{xu^2} = \frac{1}{bu} - \frac{1}{b^2} \ln \left| \frac{u}{x} \right| \quad 9 \int \frac{dx}{x^2 u^2} = -\frac{b+2ax}{b^2 xu} + \frac{2a}{b^3} \ln \left| \frac{u}{x} \right| \\
 10 \int \sqrt{u} dx &= \frac{2}{3a} u^{3/2} \quad 11 \int x\sqrt{u} dx = \frac{2(3ax-2b)}{15a^2} u^{3/2} \quad 12 \int x^2 \sqrt{u} dx = \frac{2(15a^2 x^2 - 12abx + 8b^2)}{105a^3} u^{3/2} \\
 13 \int \frac{\sqrt{u}}{x} dx &= 2\sqrt{u} + b \int \frac{dx}{x\sqrt{u}} \quad 14 \int \frac{x dx}{\sqrt{u}} = \frac{2(ax-2b)}{3a^2} \sqrt{u} \quad 15 \int \frac{x^2 dx}{\sqrt{u}} = \frac{2(3a^2 x^2 - 4abx + 8b^2)}{15a^3} \sqrt{u} \\
 16 \int \frac{dx}{x\sqrt{u}} &= \frac{1}{\sqrt{b}} \ln \left| \frac{\sqrt{u}-\sqrt{b}}{\sqrt{u}+\sqrt{b}} \right| \quad (b > 0) \quad \text{or } \frac{2}{\sqrt{-b}} \tan^{-1} \frac{\sqrt{u}}{\sqrt{-b}} \quad (b < 0) \quad 17 \int \frac{\sqrt{u}}{x^2} dx = -\frac{\sqrt{u}}{x} + \frac{a}{2} \int \frac{dx}{x\sqrt{u}} \\
 18 \int \frac{dx}{(ax+b)(cx+d)} &= \frac{1}{bc-ad} \ln \left| \frac{cx+d}{ax+b} \right| \quad 19 \int \frac{x dx}{(ax+b)(cx+d)} = \frac{1}{bc-ad} \left( \frac{b}{a} \ln|ax+b| - \frac{d}{c} \ln|cx+d| \right) \\
 20 \int \sqrt{x^2 \pm a^2} dx &= \frac{x}{2} \sqrt{x^2 \pm a^2} \pm \frac{a^2}{2} \ln|x + \sqrt{x^2 \pm a^2}| \quad 21 \int \frac{dx}{\sqrt{x^2 \pm a^2}} = \ln|x + \sqrt{x^2 \pm a^2}| \\
 22 \int \frac{\sqrt{x^2+a^2}}{x} dx &= \sqrt{x^2+a^2} - a \ln \left( \frac{a+\sqrt{x^2+a^2}}{x} \right) \quad 23 \int \frac{\sqrt{x^2-a^2}}{x} dx = \sqrt{x^2-a^2} - a \sec^{-1} \frac{x}{a} \\
 24 \int \frac{x^2 dx}{\sqrt{x^2 \pm a^2}} &= \frac{x}{2} \sqrt{x^2 \pm a^2} \mp \frac{a^2}{2} \ln|x + \sqrt{x^2 \pm a^2}| \quad 25 \int \frac{\sqrt{x^2 \pm a^2}}{x^2} dx = -\frac{\sqrt{x^2 \pm a^2}}{x} + \ln|x + \sqrt{x^2 \pm a^2}| \\
 26 \int x^2 \sqrt{x^2 \pm a^2} dx &= \frac{x}{8} (2x^2 \pm a^2) \sqrt{x^2 \pm a^2} - \frac{a^4}{8} \ln|x + \sqrt{x^2 \pm a^2}| \quad 27 \int \frac{dx}{x^2 \sqrt{x^2 \pm a^2}} = \mp \frac{\sqrt{x^2 \pm a^2}}{a^2 x} \\
 28 \int (x^2 \pm a^2)^{3/2} dx &= \frac{x}{8} (2x^2 \pm 5a^2) \sqrt{x^2 \pm a^2} + \frac{3a^4}{8} \ln|x + \sqrt{x^2 \pm a^2}| \quad 29 \int \frac{dx}{(x^2 \pm a^2)^{3/2}} = \frac{\pm x}{a^2 \sqrt{x^2 \pm a^2}} \\
 30 \int \sqrt{a^2 - x^2} dx &= \frac{x}{2} \sqrt{a^2 - x^2} + \frac{a^2}{2} \sin^{-1} \frac{x}{a} \quad 31 \int \frac{dx}{\sqrt{a^2 - x^2}} = \sin^{-1} \frac{x}{a} \quad 32 \int \frac{dx}{(a^2 - x^2)^{3/2}} = \frac{x}{a^2 \sqrt{a^2 - x^2}} \\
 33 \int \frac{\sqrt{a^2 - x^2}}{x} dx &= \sqrt{a^2 - x^2} - a \ln \left| \frac{a+\sqrt{a^2-x^2}}{x} \right| \quad 34 \int x^2 \sqrt{a^2 - x^2} dx = \frac{x}{8} (2x^2 - a^2) \sqrt{a^2 - x^2} + \frac{a^4}{8} \sin^{-1} \frac{x}{a} \\
 35 \int \frac{dx}{x^2 \sqrt{a^2 - x^2}} &= -\frac{\sqrt{a^2 - x^2}}{a^2 x} \quad 36 \int \frac{\sqrt{a^2 - x^2}}{x^2} dx = -\frac{\sqrt{a^2 - x^2}}{x} - \sin^{-1} \frac{x}{a} \quad 37 \int \frac{dx}{x \sqrt{a^2 - x^2}} = -\frac{1}{a} \ln \left| \frac{a+\sqrt{a^2-x^2}}{x} \right| \\
 38 \int \frac{x^2 dx}{\sqrt{a^2 - x^2}} &= -\frac{x}{2} \sqrt{a^2 - x^2} + \frac{a^2}{2} \sin^{-1} \frac{x}{a} \quad 39 \int (a^2 - x^2)^{3/2} dx = \frac{x}{8} (5a^2 - 2x^2) \sqrt{a^2 - x^2} + \frac{3a^4}{8} \sin^{-1} \frac{x}{a} \\
 40 \int \frac{dx}{b+c \sin ax} &= \frac{-2}{a\sqrt{b^2-c^2}} \tan^{-1} \left[ \sqrt{\frac{b-c}{b+c}} \tan \left( \frac{\pi}{4} - \frac{ax}{2} \right) \right], \quad b^2 > c^2 \quad 41 \int \frac{dx}{1+\sin ax} = -\frac{1}{a} \tan \left( \frac{\pi}{4} - \frac{ax}{2} \right) \\
 42 \int \frac{dx}{b+c \sin ax} &= \frac{-1}{a\sqrt{c^2-b^2}} \ln \left| \frac{c+b \sin ax + \sqrt{c^2-b^2} \cos ax}{b+c \sin ax} \right|, \quad b^2 < c^2 \quad 43 \int \frac{dx}{1-\sin ax} = \frac{1}{a} \tan \left( \frac{\pi}{4} + \frac{ax}{2} \right) \\
 44 \int \frac{dx}{b+c \cos ax} &= \frac{2}{a\sqrt{b^2-c^2}} \tan^{-1} \left[ \sqrt{\frac{b-c}{b+c}} \tan \frac{ax}{2} \right], \quad b^2 > c^2 \quad 45 \int \frac{dx}{1+\cos ax} = \frac{1}{a} \tan \frac{ax}{2} \\
 46 \int \frac{dx}{b+c \cos ax} &= \frac{1}{a\sqrt{c^2-b^2}} \ln \left| \frac{c+b \cos ax + \sqrt{c^2-b^2} \sin ax}{b+c \cos ax} \right|, \quad b^2 < c^2 \quad 47 \int \frac{dx}{1-\cos ax} = -\frac{1}{a} \cot \frac{ax}{2} \\
 48 \int \sin^{-1} ax dx &= x \sin^{-1} ax + \frac{1}{a} \sqrt{1-a^2 x^2} \quad 49 \int x^n \sin^{-1} ax dx = \frac{x^{n+1}}{n+1} \sin^{-1} ax - \frac{a}{n+1} \int \frac{x^{n+1} dx}{\sqrt{1-a^2 x^2}} \\
 50 \int \tan^{-1} ax dx &= x \tan^{-1} ax - \frac{1}{2a} \ln(1+a^2 x^2) \quad 51 \int x^n \tan^{-1} ax dx = \frac{x^{n+1}}{n+1} \tan^{-1} ax - \frac{a}{n+1} \int \frac{x^{n+1} dx}{1+a^2 x^2} \\
 52 \int e^{ax} dx &= \frac{e^{ax}}{a} \quad 53 \int x e^{ax} dx = \frac{e^{ax}}{a^2} (ax-1) \quad 54 \int x^2 e^{ax} dx = \frac{e^{ax}}{a^3} (a^2 x^2 - 2ax + 2) \quad (b^{ax} \text{ is } e^{a(\ln b)x}) \\
 55 \int \frac{dx}{x \ln ax} &= \ln |\ln ax| \quad \text{Not elementary: } \int e^{x^2} dx, \int e^x \ln x dx, \int \frac{dx}{\ln x}, \int \frac{e^x}{x} dx, \int \frac{\sin x}{x} dx, \int \frac{\sin^{-1} x}{x} dx
 \end{aligned}$$

## Exponentials and Logarithms

$$y = b^x \leftrightarrow x = \log_b y \quad y = e^x \leftrightarrow x = \ln y$$

$$e = \lim(1 + \frac{1}{n})^n = \sum_{n=0}^{\infty} \frac{1}{n!} = 2.71828 \dots$$

$$e^x = \lim(1 + \frac{x}{n})^n = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

$$\ln y = \int_1^y \frac{dx}{x} \quad \ln 1 = 0 \quad \ln e = 1$$

$$\ln xy = \ln x + \ln y \quad \ln x^n = n \ln x$$

$$\log_a y = (\log_a b)(\log_b y) \quad \log_a b = 1/\log_b a$$

$$e^{x+y} = e^x e^y \quad b^x = e^{x \ln b} \quad e^{\ln y} = y$$

## Vectors and Determinants

$$\mathbf{A} = a_1 \mathbf{i} + a_2 \mathbf{j} + a_3 \mathbf{k}$$

$$|\mathbf{A}|^2 = \mathbf{A} \cdot \mathbf{A} = a_1^2 + a_2^2 + a_3^2 \text{ (length squared)}$$

$$\mathbf{A} \cdot \mathbf{B} = a_1 b_1 + a_2 b_2 + a_3 b_3 = |\mathbf{A}||\mathbf{B}| \cos \theta$$

$$|\mathbf{A} \cdot \mathbf{B}| \leq |\mathbf{A}||\mathbf{B}| \text{ (Schwarz inequality: } |\cos \theta| \leq 1)$$

$$|\mathbf{A} + \mathbf{B}| \leq |\mathbf{A}| + |\mathbf{B}| \text{ (triangle inequality)}$$

$$|\mathbf{A} \times \mathbf{B}| = |\mathbf{A}||\mathbf{B}| \sin \theta \text{ (cross product)}$$

$$\mathbf{A} \times \mathbf{B} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = \mathbf{i}(a_2 b_3 - a_3 b_2) + \mathbf{j}(a_3 b_1 - a_1 b_3) + \mathbf{k}(a_1 b_2 - a_2 b_1)$$

Right hand rule  $\mathbf{i} \times \mathbf{j} = \mathbf{k}$ ,  $\mathbf{j} \times \mathbf{k} = \mathbf{i}$ ,  $\mathbf{k} \times \mathbf{i} = \mathbf{j}$

Parallelogram area =  $|a_1 b_2 - a_2 b_1| = |\text{Det}|$

Triangle area =  $\frac{1}{2}|a_1 b_2 - a_2 b_1| = \frac{1}{2}|\text{Det}|$

Box volume =  $|\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C})| = |\text{Determinant}|$

## Equations and Their Solutions

$$y' = cy \quad y_0 e^{ct}$$

$$y' = cy + s \quad y_0 e^{ct} + \frac{s}{c}(e^{ct} - 1)$$

$$y' = cy - by^2 \quad \frac{c}{b+de^{-ct}} \quad d = \frac{c-by_0}{y_0}$$

$$y'' = -\lambda^2 y \quad \cos \lambda t \text{ and } \sin \lambda t$$

$$my'' + dy' + ky = 0 \quad e^{\lambda_1 t} \text{ and } e^{\lambda_2 t} \text{ or } te^{\lambda_1 t}$$

$$y_{n+1} = ay_n \quad a^n y_0$$

$$y_{n+1} = ay_n + s \quad a^n y_0 + s \frac{a^n - 1}{a - 1}$$

## Matrices and Inverses

$Ax =$  combination of columns =  $b$

Solution  $x = A^{-1}b$  if  $A^{-1}A = I$

Least squares  $A^T A \bar{x} = A^T b$

$Ax = \lambda x$  ( $\lambda$  is an eigenvalue)

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

$$(AB)^{-1} = B^{-1}A^{-1}, (AB)^T = B^T A^T$$

$$\begin{bmatrix} \mathbf{a} & \mathbf{b} & \mathbf{c} \end{bmatrix}^{-1} = \frac{1}{D} \begin{bmatrix} \mathbf{b} \times \mathbf{c} \\ \mathbf{c} \times \mathbf{a} \\ \mathbf{a} \times \mathbf{b} \end{bmatrix}$$

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = \begin{matrix} + a_1 b_2 c_3 + a_2 b_3 c_1 + a_3 b_1 c_2 \\ - a_1 b_3 c_2 - a_2 b_1 c_3 - a_3 b_2 c_1 \end{matrix}$$

	SI Units	Symbols	From	To	Multiply by
length	meter	m	degrees	radians	.01745
mass	kilogram	kg	calories	joules	4.1868
time	second	s	BTU	joules	1055.1
current	ampere	A	foot-pounds	joules	1.3558
frequency	hertz	Hz $\sim 1/s$	feet	meters	.3048
force	newton	N $\sim \text{kg}\cdot\text{m}/\text{s}^2$	miles	km	1.609
pressure	pascal	Pa $\sim \text{N}/\text{m}^2$	feet/sec	km/hr	1.0973
energy, work	joule	J $\sim \text{N}\cdot\text{m}$	pounds	kg	.45359
power	watt	W $\sim \text{J}/\text{s}$	ounces	kg	.02835
charge	coulomb	C $\sim \text{A}\cdot\text{s}$	gallons	liters	3.785
temperature	kelvin	K	horsepower	watts	745.7
Speed of light	$c = 2.9979 \times 10^8 \text{ m/s}$		Radius at Equator	$R = 6378 \text{ km} = 3964 \text{ miles}$	
Gravity	$G = 6.6720 \times 10^{-11} \text{ Nm}^2/\text{kg}^2$		Acceleration	$g = 9.8067 \text{ m/s}^2 = 32.174 \text{ ft/s}^2$	

### Sums and Infinite Series

$$1 + x + \dots + x^{n-1} = \frac{1-x^n}{1-x}$$

$$1 + nx + \frac{n(n-1)}{2!}x^2 + \dots + x^n = (1+x)^n$$

$$1 + 2 + \dots + n = \frac{1}{2}n(n+1) \approx \frac{n^2}{2}$$

$$1^2 + 2^2 + \dots + n^2 = \frac{n(n+1)(2n+1)}{6} \approx \frac{n^3}{3}$$

$$1 + \frac{1}{2} + \dots + \frac{1}{n} \approx \ln n \rightarrow \infty \text{ (harmonic)}$$

$$1 - \frac{1}{2} + \frac{1}{3} - \dots = \ln 2 \text{ (alternating)}$$

$$1 - \frac{1}{3} + \frac{1}{5} - \dots = \frac{\pi}{4} \quad \sum \frac{1}{n^2} = \frac{\pi^2}{6} \quad \sum \frac{1}{n^4} = \frac{\pi^4}{90}$$

$$\frac{1}{1-x} = 1 + x + x^2 + \dots \text{ (geometric: } |x| < 1)$$

$$\frac{1}{(1-x)^2} = 1 + 2x + 3x^2 + \dots = \frac{d}{dx} \left( \frac{1}{1-x} \right)$$

$$\frac{1}{1+x} = 1 - x + x^2 - \dots \text{ (geometric for } -x)$$

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \dots = \int \frac{dx}{1+x}$$

$$\sin x = x - x^3/6 + x^5/120 - \dots \text{ (all } x)$$

$$\cos x = 1 - x^2/2 + x^4/24 - \dots \text{ (all } x)$$

$$e^x = 1 + x + \frac{x^2}{2!} + \dots \text{ (} e = 1 + 1 + \frac{1}{2!} + \dots)$$

$$e^{ix} = \cos x + i \sin x \text{ (Euler's formula)}$$

$$\cosh x = \frac{1}{2}(e^x + e^{-x}) = 1 + \frac{x^2}{2!} + \dots$$

$$\sinh x = \frac{1}{2}(e^x - e^{-x}) = x + \frac{x^3}{3!} + \dots$$

$$(\cos \theta + i \sin \theta)^n = \cos n\theta + i \sin n\theta$$

$$f(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \dots \text{ (Taylor)}$$

$$f(x, y) = f + x f_x + y f_y + \frac{x^2}{2!} f_{xx} + xy f_{xy} + \dots$$

### Polar and Spherical

$$x = r \cos \theta \text{ and } y = r \sin \theta$$

$$r = \sqrt{x^2 + y^2} \text{ and } \tan \theta = y/x$$

$$x + iy = r(\cos \theta + i \sin \theta) = re^{i\theta}$$

$$\text{Area } \int \frac{1}{2}r^2 d\theta \quad \text{Length } \int \sqrt{r_\theta^2 + r^2} d\theta$$

$$x = \rho \sin \phi \cos \theta, y = \rho \sin \phi \sin \theta, z = \rho \cos \phi$$

$$\text{Area } dA = dx dy = r dr d\theta = J du dv$$

$$\text{Volume } r dr d\theta dz = \rho^2 \sin \phi d\rho d\phi d\theta$$

$$\text{Stretching factor } J = \frac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} x_u & x_v \\ y_u & y_v \end{vmatrix}$$

An additional table of integrals is included just after the index.

### Area - Volume - Length - Mass - Moment

Circle  $\pi r^2$  Ellipse  $\pi ab$  Wedge of circle  $r^2\theta/2$   
 Cylinder side  $2\pi rh$  Volume  $\pi r^2 h$  Shell  $dV = 2\pi r h dr$   
 Sphere surface  $4\pi r^2$  Volume  $\frac{4}{3}\pi r^3$  Shell  $dV = 4\pi r^2 dr$   
 Cone or pyramid Volume  $\frac{1}{3}$  (base area) (height)  
 Length of curve  $\int ds = \int \sqrt{1 + (dy/dx)^2} dx$   
 Area between curves  $\int (v(x) - w(x)) dx$   
 Surface area of revolution  $\int 2\pi r ds$  ( $r = x$  or  $r = y$ )  
 Volume of revolution: Slices  $\int \pi y^2 dx$  Shells  $\int 2\pi x h dx$   
 Area of surface  $z(x, y) : \iint \sqrt{1 + z_x^2 + z_y^2} dx dy$   
 Mass  $M = \iint \rho dA$  Moment  $M_y = \iint \rho x dA$   
 $\bar{x} = M_y/M, \bar{y} = M_x/M$  Moment of Inertia  $I_y = \iint \rho x^2 dA$   
 Work  $W = \int_a^b F(x) dx = V(b) - V(a)$  Force  $F = dV/dx$

### Partial Derivatives of $z = f(x, y)$

Tangent plane  $z - z_0 = \left(\frac{\partial f}{\partial x}\right)(x - x_0) + \left(\frac{\partial f}{\partial y}\right)(y - y_0)$   
 Approximation  $\Delta z \approx \left(\frac{\partial f}{\partial x}\right)\Delta x + \left(\frac{\partial f}{\partial y}\right)\Delta y$   
 Normal  $\mathbf{N} = (f_x, f_y, -1)$  or  $(F_x, F_y, F_z)$   
 Gradient  $\nabla f = \frac{\partial f}{\partial x} \mathbf{i} + \frac{\partial f}{\partial y} \mathbf{j}$   
 Directional derivative:  $D_{\mathbf{u}} f = \nabla f \cdot \mathbf{u} = f_x u_1 + f_y u_2$   
 Chain rule:  $\frac{dz}{dt} = \frac{\partial f}{\partial x} \frac{dx}{dt} + \frac{\partial f}{\partial y} \frac{dy}{dt}$

### Vector field $\mathbf{F}(x, y, z) = M \mathbf{i} + N \mathbf{j} + P \mathbf{k}$

Work  $\int \mathbf{F} \cdot d\mathbf{R}$  Flux  $\int M dy - N dx$   
 Divergence of  $\mathbf{F} = \nabla \cdot \mathbf{F} = \frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} + \frac{\partial P}{\partial z}$   
 Curl of  $\mathbf{F} = \nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ M & N & P \end{vmatrix}$   
 Conservative  $\mathbf{F} = \nabla f =$  gradient of  $f$  if  $\text{curl } \mathbf{F} = \mathbf{0}$   
 Green's Theorem  $\oint M dx + N dy = \iint \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx dy$   
 Divergence Theorem  $\iint \mathbf{F} \cdot \mathbf{n} dS = \iiint \text{div } \mathbf{F} dV$   
 Stokes' Theorem  $\oint \mathbf{F} \cdot d\mathbf{R} = \iint (\text{curl } \mathbf{F}) \cdot \mathbf{n} dS$

MIT OpenCourseWare  
<http://ocw.mit.edu>

Resource: Calculus Online Textbook  
Gilbert Strang

The following may not correspond to a particular course on MIT OpenCourseWare, but has been provided by the author as an individual learning resource.

For information about citing these materials or our Terms of Use, visit: <http://ocw.mit.edu/terms>.