Derivative of the Sine and Cosine

This lecture shows that \( \frac{d}{dx}(\sin x) = \cos x \) and \( \frac{d}{dx}(\cos x) = -\sin x \)

We have to measure the angle \( x \) in radians 2\( \pi \) radians = full 360 degrees

All the way around the circle (2\( \pi \) radians) Length = 2\( \pi \) when the radius is 1

Part way around the circle (\( x \) radians) Length = \( x \) when the radius is 1

Slope \( \cos x \)

at \( x = 0 \) slope 1 = \( \cos 0 \)
at \( x = \pi/2 \) slope 0 = \( \cos \pi/2 \)
at \( x = \pi \) slope -1 = \( \cos \pi \)

Slope - \( \sin x \)

at \( x = 0 \) slope = 0 = - \( \sin 0 \)
at \( x = \pi/2 \) slope -1 = - \( \sin \pi/2 \)
at \( x = \pi \) slope = 0 = - \( \sin \pi \)

Problem: \( \frac{\Delta y}{\Delta x} = \frac{\sin(x + \Delta x) - \sin x}{\Delta x} \) is not as simple as \( \frac{(x + \Delta x)^2 - x^2}{\Delta x} \)

Good idea to start at \( x = 0 \) Show \( \frac{\Delta y}{\Delta x} = \frac{\sin \Delta x}{\Delta x} \) approaches 1

Draw a right triangle with angle \( \Delta x \) to see \( \sin \Delta x \leq \Delta x \)

straight piece = \( \sin \Delta x \)

curved length = \( \Delta x \)

IDEA \( \frac{\sin \Delta x}{\Delta x} < 1 \) and \( \frac{\sin \Delta x}{\Delta x} > \cos \Delta x \) will \textbf{squeeze} \( \frac{\sin \Delta x}{\Delta x} \to 1 \) as \( \Delta x \to 0 \)
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To prove \( \frac{\sin \Delta x}{\Delta x} > \cos \Delta x \) which is \( \tan \Delta x > \Delta x \)  

Go to a bigger triangle

Triangle area = \( \frac{1}{2} \tan \Delta x \) greater than
Circular area = \( \frac{\Delta x}{2\pi} \) (whole circle) = \( \frac{1}{2} \Delta \)

The squeeze \( \cos \Delta x < \frac{\sin \Delta x}{\Delta x} < 1 \) tells us that \( \frac{\sin \Delta x}{\Delta x} \) approaches 1

\[
\frac{(\sin \Delta x)^2}{(\Delta x)^2} < 1 \text{ means } \frac{(1-\cos \Delta x)}{\Delta x}(1+\cos \Delta x) < \Delta x
\]

So \( \frac{1-\cos \Delta x}{\Delta x} \to 0 \)  
Cosine curve has slope = 0

For the slope at other \( x \) remember a formula from trigonometry:

\[
\sin(x + \Delta x) = \sin x \cos \Delta x + \cos x \sin \Delta x
\]

We want \( \Delta y = \sin(x + \Delta x) - \sin x \)  
Divide that by \( \Delta x \)

\[
\frac{\Delta y}{\Delta x} = (\sin x) \left( \frac{\cos \Delta x - 1}{\Delta x} \right) + (\cos x) \left( \frac{\sin \Delta x}{\Delta x} \right)
\]

Now let \( \Delta x \to 0 \)

In the limit \( \frac{dy}{dx} = (\sin x)(0) + (\cos x)(1) = \cos x \) = Derivative of \( \sin x \)

For \( y = \cos x \) the formula for \( \cos(x + \Delta x) \) leads similarly to \( \frac{dy}{dx} = -\sin x \)

Practice Questions

1. What is the slope of \( y = \sin x \) at \( x = \pi \) and at \( x = 2\pi \)?
2. What is the slope of \( y = \cos x \) at \( x = \pi/2 \) and \( x = 3\pi/2 \)?
3. The slope of \( (\sin x)^2 \) is \( 2 \sin x \cos x \). The slope of \( (\cos x)^2 \) is \( -2 \cos x \sin x \).  
   Combined, the slope of \( (\sin x)^2 + (\cos x)^2 \) is zero. Why is this true?
4. What is the second derivative of \( y = \sin x \) (derivative of the derivative)?
5. At what angle \( x \) does \( y = \sin x + \cos x \) have zero slope?
6. Here are amazing infinite series for \( \sin x \) and \( \cos x \). \( e^{ix} = \cos x + i \sin x \)

\[
\sin x = \frac{x}{1} - \frac{x^3}{3 \cdot 2 \cdot 1} + \frac{x^5}{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1} - \cdots \quad \text{(odd powers of } x) \\
\cos x = 1 - \frac{x^2}{2 \cdot 1} + \frac{x^4}{4 \cdot 3 \cdot 2 \cdot 1} - \cdots \quad \text{(even powers of } x)
\]

7. Take the derivative of the sine series to see the cosine series.

8. Take the derivative of the cosine series to see minus the sine series.

9. Those series tell us that for small angles \( \sin x \approx x \) and \( \cos x \approx 1 - \frac{1}{2}x^2 \). With these approximations check that \((\sin x)^2 + (\cos x)^2\) is close to 1.
Resource: Highlights of Calculus
Gilbert Strang

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