Differential Equations of Growth

\[ \frac{dy}{dt} = cy \quad \text{Complete solution} \quad y(t) = Ae^{ct} \quad \text{for any } A \]

Starting from \( y(0) \) \( y(t) = y(0)e^{ct} \) \( A = y(0) \)

Now include a constant source term \( s \) \quad This gives a new equation

\[ \frac{dy}{dt} = cy + s \quad s > 0 \text{ is saving}, \quad s < 0 \text{ is spending}, \quad cy \text{ is interest} \]

Complete solution \( y(t) = -\frac{s}{c} + Ae^{ct} \) (any \( A \) gives a solution)

\( y = -\frac{s}{c} \) is a constant solution with \( cy + s = 0 \) and \( \frac{dy}{dt} = 0 \) and \( A = 0 \)

For that solution, the spending \( s \) exactly balances the income \( cy \)

Choose \( A \) to start from \( y(0) \) at \( t = 0 \) \( y(t) = -\frac{s}{c} + \left( y(0) + \frac{s}{c} \right)e^{ct} \)

Now add a nonlinear term \( sP^2 \) coming from competition

\( P(t) = \text{world population at time } t \) (for example) follows a new equation

\[ \frac{dP}{dt} = cP - sP^2 \quad c = \text{birth rate minus death rate} \]

“LOGISTIC EQN” \( P^2 \) since each person competes with each person

To bring back a linear equation set \( y = \frac{1}{P} \)

Then \( \frac{dy}{dt} = \frac{-dP/dt}{P^2} = \frac{(-cP + sP^2)}{P^2} = -\frac{c}{P} + s = -cy + s \)

\( y = 1/P \) produced our linear equation (no \( y^2 \)) with \( -c \) not \( +c \)

\( y(t) = \frac{s}{c} + Ae^{-ct} = \frac{s}{c} + \left( y(0) - \frac{s}{c} \right)e^{-ct} = \text{old solution with change to } -c \)

At \( t = 0 \) we correctly get \( y(0) \) \quad CORRECT START

As \( t \to \infty \) and \( e^{-ct} \to 0 \) we get \( y(\infty) = \frac{s}{c} \) and \( P(\infty) = \frac{c}{s} \)

The population \( P(t) \) increases along an \textbf{S-curve} approaching \( \frac{c}{s} \)
Differential Equations of Growth

\[ P = \frac{c}{2s} \text{ has } P'' = 0 \quad \text{Inflection point} \quad \text{Bending changes from up to down} \]

CHECK \[ \frac{d^2 P}{dt^2} = \frac{d}{dt} \left( cP - sP^2 \right) = (c - 2sP) \frac{dP}{dt} = 0 \text{ at } P = \frac{c}{2s} \]

World population approaches the limit \( \frac{c}{s} \approx 12 \text{ billion (FOR THIS MODEL!)} \)

Population now \( \approx 7 \text{ billion} \quad \text{Try Google for “World population”} \)

Practice Questions

\[ \frac{dy}{dt} = cy - s \text{ has } s = \text{spending rate not savings rate (with minus sign)} \]

1. The constant solution is \( y = \) when \( \frac{dy}{dt} = 0 \)

In that case interest income balances spending: \( cy = s \)

2. The complete solution is \( y(t) = \frac{s}{c} + Ae^{ct} \). Why is \( A = y(0) - \frac{s}{c} \)?

3. If you start with \( y(0) > \frac{s}{c} \) why does wealth approach \( \infty \)?

   If you start with \( y(0) < \frac{s}{c} \) why does wealth approach \(-\infty \)?

4. The complete solution to \( \frac{dy}{dt} = s \) is \( y(t) = st + A \)

What solution \( y(t) \) starts from \( y(0) \) at \( t = 0 \)?

5. If \( \frac{dP}{dt} = -sP^2 \) and \( y = \frac{1}{P} \) explain why \( \frac{dy}{dt} = s \)

Pure competition. Show that \( P(t) \to 0 \) as \( t \to \infty \)

6. If \( \frac{dP}{dt} = cP - sP^4 \) find a linear equation for \( y = \frac{1}{P^3} \)
Resource: Highlights of Calculus
Gilbert Strang

The following may not correspond to a particular course on MIT OpenCourseWare, but has been provided by the author as an individual learning resource.

For information about citing these materials or our Terms of Use, visit: http://ocw.mit.edu/terms.