Inverse Functions and Logarithms

A function assigns an output \( y = f(x) \) to each input \( x \).

A one-to-one function has different outputs \( y \) for different inputs \( x \).

For the inverse function the input is \( y \) and the output is \( x = f^{-1}(y) \).

Example: If \( y = f(x) = x^5 \) then \( x = f^{-1}(y) = y^{\frac{1}{5}} \).

KEY: If \( y = ax + b \) then solve for \( x = \frac{y - b}{a} \) = inverse function.

Notice that \( x = f^{-1}(f(x)) \) and \( y = f(f^{-1}(y)) \).

The chain rule will connect the derivatives of \( f^{-1} \) and \( f \).

The great function of calculus is \( y = e^x \).

Its inverse function is the “natural logarithm” \( x = \ln y \).

Remember that \( x \) is the exponent in \( y = e^x \).

The rule \( e^x e^x = e^{x+x} \) tells us that \( \ln(yY) = \ln y + \ln Y \).

Add logarithms because you add exponents: \( \ln(e^2 e^3) = 5 \).

\( (e^x)^n = e^{nx} \) (multiply exponent) tells us that \( \ln(y^n) = n \ln y \).

We can change from base \( e \) to base 10: New function \( y = 10^x \).

The inverse function is the logarithm to base 10. Call it log: \( x = \log y \).

Then \( \log 100 = 2 \) and \( \log \frac{1}{100} = -2 \) and \( \log 1 = 0 \).

We will soon find the beautiful derivative of \( \ln y \): \( \frac{d}{dy} (\ln y) = \frac{1}{y} \).

You can change letters to write that as \( \frac{d}{dx} (\ln x) = \frac{1}{x} \).
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Practice Questions

1. What is \( x = f^{-1}(y) \) if \( y = 50x \) ?
2. What is \( x = f^{-1}(y) \) if \( y = x^4 \) ? Why do we keep \( x \geq 0 \) ?
3. Draw a graph of an increasing function \( y = f(x) \). This has different outputs \( y \) for different \( x \). Flip the graph (switch the axes) to see \( x = f^{-1}(y) \)
4. This graph has the same \( y \) from two \( x \)'s. There is no \( f^{-1}(y) \)

5. The natural logarithm of \( y = 1/e \) is \( \ln(e^{-1}) = ? \) What is \( \ln(\sqrt{e}) \) ?
6. The natural logarithm of \( y = 1 \) is \( \ln 1 = ? \) and also base 10 has \( \log 1 = ? \)
7. The natural logarithm of \( (e^2)^{50} \) is ? The base 10 logarithm of \( (10^2)^{50} \) is ?
8. I believe that \( \ln y = (\ln 10)(\log y) \) because we can write \( y \) in two ways \( y = e^{\ln y} \) and also \( y = 10^{\log y} = e^{(\ln 10)(\log y)} \). Explain those last steps.
9. Change from base \( e \) and base 10 to base 2. Now \( y = 2^x \) means \( x = \log_2 y \). What are \( \log_2 32 \) and \( \log_2 2 \)? Why is \( \log_2(e) > 1 \) ?