Linear Approximation and Newton’s Method

Start at \( x = a \) with known \( f(a) \) = height and \( f'(a) \) = slope

**KEY IDEA** \( f'(a) \approx \frac{f(x) - f(a)}{x - a} \) when \( x \) is near \( a \)

Tangent line has slope \( f'(a) \)
Solve for \( f(x) \)
\[ f(x) \approx f(a) + (x - a) f'(a) \]

\( \approx \) means “approximately”
curve \( \approx \) line near \( x = a \)

**Examples of linear approximation to \( f(x) \)**

1. \( f(x) = e^x \) \( f(0) = e^0 = 1 \) and \( f'(0) = e^0 = 1 \) are known at \( a = 0 \)
   Follow the tangent line \( e^x \approx 1 + (x - 0)1 = 1 + x \)
   \( 1 + x \) is the linear part of the series for \( e^x \)

2. \( f(x) = x^{10} \) and \( f'(x) = 10x^9 \) \( f(1) = 1 \) and \( f'(1) = 10 \) known at \( a = 1 \)
   Follow the tangent line \( x^{10} \approx 1 + (x - 1)10 \) near \( x = 1 \)
   Take \( x = 1.1 \) \( (1.1)^{10} \) is approximately \( 1 + 1 = 2 \)

**Newton’s Method** (looking for \( x \) to nearly solve \( f(x) = 0 \))

Go back to \( f'(a) \approx \frac{f(x) - f(a)}{x - a} \)
\( f(a) \) and \( f'(a) \) are again known

Solve for \( x \) when \( f(x) = 0 \)
\[ x - a \approx \frac{f(a)}{f'(a)} \] Newton \( x \)
Line crossing near curve crossing
Examples of Newton’s Method

Solve \( f(x) = x^2 - 1.2 = 0 \)

1. \( a = 1 \) gives \( f(a) = 1 - 1.2 = -0.2 \) and \( f'(a) = 2a = 2 \)
   Tangent line hits 0 at \( x = 1 = -\left(\frac{-0.2}{2}\right) \) Newton’s \( x \) will be 1.1

2. For a better \( x \), Newton starts again from that point \( a = 1.1 \)
   Now \( f(a) = 1.1^2 - 1.2 = .01 \) and \( f'(a) = 2a = 2.2 \)
   The new tangent line has \( x = 1.1 = \frac{0.1}{2.2} \) For this \( x \), \( x^2 \) is very close to 1.2

---

Practice Questions

1. The graph of \( y = f(a) + (x - a)f'(a) \) is a straight _____
   At \( x = a \) the height is \( y = _____ \)
   At \( x = a \) the slope is \( \frac{dy}{dx} = _____ \)
   This graph is t _____ t to the graph of \( f(x) \) at \( x = a \)
   For \( f(x) = x^2 \) at \( a = 3 \) this linear approximation is \( y = _____ \)

2. \( y = f(a) + (x - a)f'(a) \) has \( y = 0 \) when \( x - a = _____ \)
   Instead of the curve \( f(x) \) crossing 0, Newton has tangent line \( y \) crossing 0
   \( f(x) = x^3 - 8.12 \) at \( a = 2 \) has \( f(a) = _____ \) and \( f'(a) = 3a^2 = _____ \)
   Newton’s method gives \( x - 2 = \frac{f(a)}{f'(a)} = _____ \)
   This Newton \( x = 2.01 \) nearly has \( x^3 = 8.12 \). It actually has \( (2.01)^3 = _____ \).