Calculus Revisited
Part 1
A Self-Study Course

Lecture Notes

Center for Advanced Engineering Study

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CALCULUS REVISITED
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Area under a Curve

Physical Interpretation
� = e², 0 ≤ e ≤ 1

Area distance < ⁿ²

Zero's Paradox
Tortoise and the Hare

(a) Discrete Limit
Area is defined as an "endless" sum of areas of rectangles.
How big is an "infinite" sum?

1 + \(\frac{1}{2}\) + \(\frac{1}{4}\) + \(\frac{1}{8}\) + \(\frac{1}{16}\) + ... = 2

\(\frac{x}{2}\) = \(\frac{1}{2}\)
\(\frac{x}{2}\) = \(x - 1\)
\(x = 2\)

Areas and Rates of Change

Tangent line passes through "two consecutive points" on the curve

Two-lim Concepts

Functions (Sets)
Limits
Derivatives (Rate of change)
Integrals (Area under curves)

Applications
More "elaborate" functions
More sophisticated techniques
Infinite Series
1.010 Analytic Geometry

37 min.

**Analytic Geometry**

**Profit**

- **Points and Ordered Pairs**

- **Graphs:**
  - Graph showing ordered pairs (x, y) and their relationship to the x-axis and y-axis.

- **Equations and Formulas:**
  - \( a = 16c^2 \)
  - \( V = \pi r^2 h \)
  - \( C(h) = 2\pi rh \)
  - \( (3, 2) \rightarrow 12\pi \)
  - \( (3, 2) \rightarrow 18\pi \)

- **Inequality:**
  - \( a^2 + 2ab + b^2 \) > \( a^2 + b^2 + b^2 \)
\[ S = \{(x, y) : x^2 + y^2 = 25\} \]

- \((3, 4) \in S\)
- \((1, 3) \notin S\)

**Straight line**

- \(m = 3\)
- \(y = 3x - 1\)

**Interpolation**

- \(\log_2 3 = 0.301\)
- \(\log_4 0.602 = 0.417\)

**Equations of lines**

- Line \((l)\) through \((2, 5)\) and \((0, -1)\)

**Simultaneous equations**

\[ \begin{align*}
  y &= 5x - 1 \\
  5 &= 2x + 1
\end{align*} \]

- \(3x - 1 = 2 + 1\)
- \(2x = 2\)
- \(x = 1\)
- \((1, 2)\)

**Example:**

- \(m = 3\)
- \(y = 3x - 1\)
Functions:
\[ f: A \to B \text{ means that } f \text{ is a rule which assigns to each } a \in A \text{ an element } b \in B. \]

**A** = domain of \( f \)

\[ f = \{ (a, b) | a \in A, b \in B \} \]

**B** = range of \( f \)

\[ f(A) = \{ b | \exists a \in A : f(a) = b \} \]

**C** = image of \( f \)

\[ f(A) = f(A) \]

**Onto Functions**

Range = Image

Example:

\[ A = \{1, 2, 3\} \]

\[ f(a) = 4a, a \in A \]

\[ f(1) = 4, f(2) = 8 \]

\[ f(3) = 12 \]

\[ B = \{4, 8, 12\} \]

\[ f: A \to B \]

**One-to-One Functions**

\[ f(a_1) = f(a_2) \Rightarrow a_1 = a_2 \]

**Real Variable**

\[ s = 16t^2 \]

\[ v = \frac{ds}{dt} = 32t \]

\[ s = \begin{cases} 0, & t \leq 0 \\ 16t^2 - 0.36, & 0 < t \leq \frac{1}{4} \\ 4.00, & t > \frac{1}{4} \end{cases} \]
Inverse Functions

"Switch in Emphasis"

Example

\[ y = f(x); \quad x = f'(y) \]

\[ y = \frac{1}{x}; \quad b = \frac{5}{2} \]

\[ y = \sin x; \quad x = \sin y \]

\[ f(x) = \frac{1}{x}; \quad f'(x) = \frac{1}{x^2} \]

\[ f(x) = \ln x; \quad f'(x) = \frac{1}{x} \]

\[ f(x) = \sqrt{x}; \quad f'(x) = \frac{1}{2\sqrt{x}} \]

\[ f(x) = e^x; \quad f'(x) = e^x \]

\[ f(x) = \ln x; \quad f'(x) = \frac{1}{x} \]

\[ f(x) = \cos x; \quad f'(x) = -\sin x \]

\[ f(x) = \tan x; \quad f'(x) = \sec^2 x \]

\[ f(x) = \sec x; \quad f'(x) = \sec x \tan x \]

\[ f(x) = \csc x; \quad f'(x) = -\csc x \cot x \]

\[ f(x) = \cot x; \quad f'(x) = -\csc^2 x \]

\[ f(x) = \sin^{-1} x; \quad f'(x) = \frac{1}{\sqrt{1-x^2}} \]

\[ f(x) = \cos^{-1} x; \quad f'(x) = -\frac{1}{\sqrt{1-x^2}} \]

\[ f(x) = \tan^{-1} x; \quad f'(x) = \frac{1}{1+x^2} \]

\[ f(x) = \sec^{-1} x; \quad f'(x) = \frac{1}{x\sqrt{x^2-1}} \]

\[ f(x) = \csc^{-1} x; \quad f'(x) = -\frac{1}{x\sqrt{x^2-1}} \]

\[ f(x) = \cot^{-1} x; \quad f'(x) = -\frac{1}{1+x^2} \]

[Graphs and diagrams showing various functions and their derivatives]
Derivatives and Limits

How fast does the ball fall when \( t = 1 \)?

Ave speed from \( t = 1 \) to \( t = 2 \):

\[
\frac{A(2) - A(1)}{2-1} = \frac{64 - 16}{1} = 48 \text{ ft/sec}
\]

Ave speed from \( t = 1 \) to \( t = 1+t \):

\[
\frac{A(1+t) - A(1)}{1+t - 1} = \frac{16(1+t)^3 - 16}{t} = \frac{32t + 16t^2}{t}
\]

"Appears" that speed is 32 ft/sec when \( t = 1 \)

I.e.,

\[
\lim_{h \to 0} \frac{A(t+h) - A(t)}{h} = \lim_{h \to 0} \frac{32t + 16t^2}{h} = \frac{32t + 16t^2}{h}
\]

By this approach,

if \( s = 16t^2 \) then at

time \( t = t \), \( v = 32t \),

or: \( u = 32t \),

where (instantaneous)

speed has been defined

as a limit of average

speeds.
\[ f(x + h) - f(x) \]

\[ \lim_{h \to 0} \frac{f(x + h) - f(x)}{h} \]

\[ f(x) = \lim_{x \to a} \frac{x^2 - a^2}{x - a} \]

\[ \lim_{x \to a} \frac{x^2 - a^2}{x - a} = \lim_{x \to a} \frac{(x - a)(x + a)}{x - a} \]

\[ x^2 - a^2 = (x - a)(x + a) \]

\[ \lim_{x \to a} \frac{x^2 - a^2}{x - a} = \lim_{x \to a} (x + a) = a + a = 2a \]

\[ \lim_{x \to a} f(x) = L \text{ means } \lim_{x \to a} \frac{f(x) - L}{x - a} = 0 \]

\[ \text{Given } \epsilon > 0 \text{ we can find } S > 0 \text{ such that } 0 < |x - a| < S \Rightarrow |f(x) - L| < \epsilon \]

\[ \lim_{x \to a} \frac{x^2 - 2x - 3}{x^2 - x + 3} = \frac{1}{2} \]

\[ \text{If } x \to a, \text{ then } x^2 \to a^2 \]

\[ \left( 1 + \frac{\sqrt{c}}{c} \right)^2 - 2 \left( 1 + \frac{\sqrt{c}}{c} \right) = \left( 1 + \frac{\sqrt{c}}{c} + \frac{\sqrt{c}}{c} \right) - 2 \left( 1 + \frac{\sqrt{c}}{c} \right) = \frac{3c}{c} = 3 \epsilon \]
**1.040 Limits: A More Rigorous Approach**

**46 min.**

- **Theorem**
  \[
  \lim_{x \to a} f(x) = L \\
  \text{for each } \epsilon > 0, \text{can find } \delta > 0 \text{ such that} \\
  0 < |x - a| < \delta \implies |f(x) - L| < \epsilon
  \]

- **Theorem**
  \[
  \lim_{x \to a} c = c \\
  \text{let } f(x) = c \\
  \text{then } \lim_{x \to a} f(x) = c \\
  \text{Given } \epsilon > 0 \text{ we must find } \delta > 0 \text{ such that} \\
  0 < |x - a| < \delta \implies |c - c| < \epsilon
  \]

- **Theorem**
  \[
  \lim_{x \to a} x = a \\
  \text{there exists } \delta > 0 \text{ such that} \\
  0 < |x - a| < \delta \implies |x - a| < \epsilon
  \]

- **Theorem**
  \[
  \lim_{x \to a} \left[ f(x) + g(x) \right] = \lim_{x \to a} f(x) + \lim_{x \to a} g(x) \\
  \text{let } h(x) = f(x) + g(x) \\
  \text{To prove } \lim_{x \to a} h(x) = L_1 + L_2
  \]

\[\text{for each } \epsilon > 0, \text{can find } \delta > 0 \text{ such that} \\
0 < |x - a| < \delta \implies |h(x) - (L_1 + L_2)| < \epsilon\]
Given $\epsilon > 0$, let $\delta = \frac{\epsilon}{4}$. Can find $\delta > 0$ such that

$$0 < x - \delta \leq x + \delta \rightarrow |f(x) - L_1 - L_2| \leq \epsilon \left| \left( \frac{1}{x^2} - \frac{1}{x^3} \right) \right|$$

Let $\delta = \min \{ \delta_1, \delta_2 \}$

$$0 < x - \delta \leq x + \delta \rightarrow \left| \frac{f(x) - L_1}{x^3} \right| \leq \epsilon \left| \frac{1}{x^2} \right|$$

$$\lim_{x \to 3} f(x) = L_1$$

Example:

$$\left| \lim_{x \to 3} x^2 - 7x + 10 \right| = \left| \lim_{x \to 3} (x-3)(x-1) \right|$$

$$\lim_{x \to 3} \left( x^2 - 7x + 10 \right) = \lim_{x \to 3} \left( x-3 \right) \left( x-1 \right) = 0$$

$$\lim_{x \to 3} (x-3) = 0$$

$$(3)^2 - 7(3) + 10 = 0$$

$$30$$
Mathematical Induction

\[
\lim_{x \to a} [f(x) + g(x)] = \lim_{x \to a} f(x) + \lim_{x \to a} g(x)
\]

How about

\[
\lim_{x \to a} [f(x) + g(x) + h(x)]
\]

\[
\lim_{x \to a} f(x) + \lim_{x \to a} g(x) + \lim_{x \to a} h(x)
\]

\[
\frac{1}{2} + \frac{2}{3} + \frac{3}{4} + \frac{4}{5} = 10
\]

\[
\frac{2}{3} \div 5 = \frac{1}{15}
\]

odd + odd = ? even

Positive - Positive = ?

3 - 5 = ?
Suppose
\[ \lim_{x \to a} f(x) + g(x) = \lim_{x \to a} f(x) + \lim_{x \to a} g(x) \]
\[ \lim_{x \to a} [f(x) + g(x)] = \lim_{x \to a} f(x) + \lim_{x \to a} g(x) \]

**Mathematical Induction.**
1. Show Conjecture True For \( n = 1 \)
2. Prove that truth for \( n = k \) implies the truth for \( n = k + 1 \)

\[
\begin{align*}
1 + 2 + \cdots + k &= \frac{n(n+1)}{2} \\
1 + 2 + \cdots + k + k + 1 &= \frac{k(k+1)}{2} + k + 1 \\
&= \frac{k(k+1)+2(k+1)}{2} \\
&= \frac{(k+1)(k+2)}{2} \\
\end{align*}
\]

**Prime Factors**

\[
P(41) = \text{prime}
\]

\[
\begin{align*}
P(41) &= 41 - 41 + 41 \\
&= 1 \times 41 \times 41 \\
\end{align*}
\]

**Unique Factorization Theorem**

- 2 = 2
- 3 = 3
- 4 = 2 \times 2
- 5 = 5
- 6 = 2 \times 3
- 7 = 7
- 8 = 2 \times 2 \times 2
- 9 = 3 \times 3
- 10 = 2 \times 5
- 11 = 11
- 12 = 2 \times 2 \times 3

Numbers factors considerably different than \( n \)

\[
\begin{align*}
59 &= 2 \times 3 \times 5 \times 6 \\
61 &= 1 \times 2 \times 3 \times 5 \times 6 \\
\end{align*}
\]
2.010 Derivatives of Some Simple Functions

**Derivatives of Some Simple Functions**

\[
f'(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}
\]

**Generalization**

Let \( f(x) = x^n \)

\[
f(x+\Delta x) - f(x) = (x+\Delta x)^n - x^n = n x^{n-1} \Delta x + O(\Delta x^2)
\]

\[
f'(x) = n x^{n-1}
\]

**Proof**

\[
h(x+\Delta x) = f(x+\Delta x) + g(x+\Delta x)
\]

Define \( h \) by

\[
h(x) = f(x) + g(x)
\]

Then:

\[
h(x) = f(x) + g(x)
\]

\[
. . . . . .
\]

\[
h'(x) = f'(x) + g'(x)
\]
**In fact:**

\[
h(x) = \frac{f(x)g(x)}{x}
\]

then:

\[
h(x) = \frac{f(x)g(x)}{x} - \frac{f(x)g(x)}{x}
\]

\[
= f(x)g(x) - \frac{g(x)}{x} - f(x)g(x)
\]

\[
= f(x)g(x) - \frac{g(x)}{x} - f(x)g(x)
\]

**Summary**

The basic definition:

\[
f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}
\]

never changes. But it can be manipulated to yield "convenient" "recipes".

**Note:**

It is not true always that \( \lim_{x \to a} \frac{g(x)}{x} = \frac{g(a)}{a} \)

For example, let \( g(x) = \frac{x^2 - 1}{x - 1} \)

**For a quotient:**

we can show that if \( h(x) = \frac{g(x)}{x} \), then:

\[
\lim_{x \to a} \frac{g(x)}{x} = \lim_{x \to a} \frac{g(x) - g(a)}{x - a}
\]

In our present case:

\[
\lim_{x \to a} \frac{g(x)}{x} = \lim_{x \to a} \frac{g(x) - g(a)}{x - a}
\]

**Example:**

\( f(x) = \frac{1}{x^n} \), \( n \) positive integer

\[
\lim_{x \to 0} \frac{f(x)}{x} = \lim_{x \to 0} \frac{1}{x} = \frac{1}{x}
\]

\[
f'(x) = x^{n-1} \frac{d}{dx}
\]

\[
\lim_{x \to 0} f'(x) = \lim_{x \to 0} x^{n-1} = 0
\]

\[
\lim_{x \to a} g(x) = g(a)
\]
Approximations and Infinitesimals

\[ y = x^2 \]

\[ \Delta y = (x + \Delta x)^2 - x^2 = 2x \Delta x + \Delta x^2 \]

\[ \Delta y_{\text{inf}} = 2x \Delta x \]

\[ \Delta y_{\text{def}} = \Delta y - \Delta y_{\text{inf}} = \Delta x^2 \]

\[ \frac{\Delta y}{\Delta x} = 2x + \Delta x \]

\[ \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \frac{dy}{dx} = 2x \]

\[ \frac{\Delta y}{\Delta x} = \frac{dy}{dx} + \frac{\Delta y}{\Delta x} \]

\[ \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = 0 \]

\[ y = x^2 \]

\[ \Delta y = (x + \Delta x)^2 - x^2 = 2x \Delta x + \Delta x^2 \]

\[ \Delta y_{\text{inf}} = 2x \Delta x \]

\[ \Delta y_{\text{def}} = \Delta y - \Delta y_{\text{inf}} = \Delta x^2 \]

\[ \frac{\Delta y}{\Delta x} = 2x + \Delta x \]

\[ \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \frac{dy}{dx} = 2x \]

\[ \frac{\Delta y}{\Delta x} = \frac{dy}{dx} + \frac{\Delta y}{\Delta x} \]

\[ \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = 0 \]
\[ \Delta y = \frac{dy}{dx} \Delta x + k \Delta y \]

\[ \Delta y = \Delta \frac{dy}{dx} (a_2) \]

where \( \lim_{k \to 0} \frac{dy}{dx} \)

Example:
\[ y = x^2 \]
\[ \frac{dy}{dx} = 2x \]
\[ \Delta y = 3 \Delta x \]
\[ \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = 5 \]

\[ f(x_0 + \Delta x) - f(x) \]

where \( \lim_{\Delta x \to 0} k \Delta x \)

\[ \frac{dy}{dx} \]

is not \( \frac{y_0}{x_0} \)

is one symbol. It is not \( \frac{dy}{dx} \)
Composite Functions and the Chain Rule

\[
\frac{dy}{dt} = \left( \frac{dy}{dx} \right) \left( \frac{dx}{dt} \right)
\]

\[
\lim_{k \to 0} \frac{\Delta y}{\Delta x} = \frac{dy}{dx}
\]

Example

Find \( \frac{dy}{dx} \) if

\[
y = z^2 \quad \text{and} \quad z = x^2 + 1
\]

\[
\frac{dy}{dx} = \frac{dy}{dz} \cdot \frac{dz}{dx} = 2z \cdot 2x = 2x^2 + 2
\]

\[
\frac{dy}{dt} = \frac{dy}{dz} \cdot \frac{dz}{dt} = 2z \cdot \frac{dx}{dt} = 2x \cdot \frac{dx}{dt}
\]

\[
\frac{dy}{dt} = \frac{dy}{dz} \cdot \frac{dz}{dt} = 2z \cdot \frac{dx}{dt}
\]
Differentiation of Inverse Functions

Find \( \frac{dy}{dx} \) if \( y = \frac{1}{x} \)

\[ \frac{dy}{dx} = \frac{-1}{x^2} \]
1. Proof vs Intuition

Statement: Reason

2. How likely is it that $f(x) = \lim_{x \to a} f(x)$ is 1 - 1? 

Local vs Global

$f'(c) \neq 0$
Implicit Differentiation

\[ \frac{dy}{dx} \text{ is } \frac{f'(x)}{f(x)} \]

\[ x^2 + y^2 = 25 \]

Assume \( y > 0 \)

such that

\[ x + y = 25 \]

\[ 2x + 2y \frac{dy}{dx} = 0 \]

\[ \frac{dy}{dx} = -\frac{x}{y} \]

\[ y_1(x) = \sqrt{25 - x^2} \]

\[ y_2(x) = -\sqrt{25 - x^2} \]

\[ \frac{dy_1}{dx} = \frac{x}{\sqrt{25 - x^2}} \]

\[ \frac{dy_2}{dx} = -\frac{x}{\sqrt{25 - x^2}} \]

\[ \frac{dy}{dx} = \frac{1}{y} \left( \frac{x}{\sqrt{25 - x^2}} - \frac{x}{\sqrt{25 - x^2}} \right) \]

\[ \frac{dy}{dx} = \frac{x}{y} \]
Aside:

\[ y = 2x + 8 \quad \text{if} \quad x \text{ is an integer} \]
\[ y^3 = z \]

\[ 8y^3 + 6x^2y^2 + 4x^2y + 6y^2x = 0 \]
\[ \frac{dy}{dx} = -\frac{y(x^2 + 6y)}{x^2 + 6y^3} \]
\[ \frac{dx}{dy} = -\frac{x(y^2 + 6x)}{y^2 + 6x^3} \]
\[ 4y^3 + 6x^2y = 0 \]
\[ (x^3 + y^3 + 6) = 0 \]

Find the equation of the line tangent to \( x^2 + xy + y^2 = 3 \) at the point \((1,1)\).

\[ 2x + 2y + y^2 = 6 \]
\[ 2x + y = 3 \]
\[ y = 3 - 2x \]
\[ z = x^2 + 6y + 3 \]

\[ \begin{align*}
\text{At} (1,1), & \quad \frac{dz}{dt} = 5 \\
2x + 2y + y^2 &= 6 \\
2x + y &= 3 \\
y &= 3 - 2x \\
\end{align*} \]

Related Rates

A particle moves along the curve \( x^2 + y^2 = 25 \) \((x, y \text{ in feet})\).

At \((3,4)\), \( \frac{dx}{dt} = 8 \text{ feet/sec} \).

Find \( \frac{dy}{dt} \).

\[ \begin{align*}
\frac{dx}{dt} &= -\frac{3}{10} \text{ ft/sec} \\
\frac{dy}{dt} &= -6 \text{ ft/sec} \\
\end{align*} \]

Assume \( z \) near \((2, 0)\) and \( y \) are differentiable functions of \( t \). Then:

\[ x^2 + y^2 = 25 \to \]
\[ 2x \frac{dx}{dt} + 2y \frac{dy}{dt} = 0 \]
\[ \frac{dy}{dt} = -\frac{x}{y} \frac{dx}{dt} \]

\[ -6 \text{ ft/sec} \]
Continuity

Does \( \lim_{x \to a} f(x) = f(a) \) ?

(i) \( f(a) \) must be defined. E.g., let \( f(x) = \frac{x^2 - 1}{x - 1} \).

Then:
\[
\lim_{x \to 1} f(x) = 2
\]

But \( f(1) = \frac{0}{0} \) which is undefined.

Pictorially:
\[
\begin{align*}
&\frac{x^2 - 1}{x - 1} = x + 1 \text{ except when } x = 1 \\
&g(x) = \begin{cases} 
\frac{x^2 - 1}{x - 1} & \text{if } x \neq 1 \\
2 & \text{if } x = 1
\end{cases}
\end{align*}
\]

(2) \( f(x) \) is "near" \( f(a) \)
when \( x \) is near \( a \).
E.g., curve \( y = f(x) \) is "unbroken" in a neighborhood of \( x = a \).

Definition:

(i) \( f \) is called continuous at \( x = a \) if \( \lim_{x \to a} f(x) = f(a) \).

(ii) \( f \) is called continuous on the interval \( I \) if \( \lim_{x \to a} f(x) = f(a) \)
for each \( a \) in \( I \).
**Geometric Ideas**

1. Cont. functions assume their max. and min. values on any closed interval.
   
   ![Graph of a function with a max. and min. value indicated.]

2. \( f(x) = \frac{1}{x^2} \) at \( x = 0 \).

**Analytic Ideas**

1. \( f, g \) cont. at \( x = a \).
   \[ h(x) = f(x) + g(x) \]
   \[ \lim_{x \to a} h(x) = \lim_{x \to a} [f(x) + g(x)] = \lim_{x \to a} f(x) + \lim_{x \to a} g(x) = h(a) + g(a) \]

2. Sum of two continuous functions is a continuous function.

**Intermediate Value Theorem**

If \( f \) cont. on \( [a, b] \), \( f(a) < f(b) \)

Let \( m \) be such that \( f(a) < m < f(b) \)

Then we can find \( c \in (a, b) \)

such that \( f(c) = m \).
2.060  Curve Plotting  31 min.

Curve Plotting
(with and without
calculation)

$$f(x) = f(-x)$$

Aside:
Even and Odd Functions

$$f(x) = f(-x) \rightarrow f \text{ even}$$

$$f(x) = -f(-x) \rightarrow f \text{ odd}$$

Examples:

$$y = x^2 + 2$$
$$y = (x^2) + 2$$
$$f(x) = f(-x)$$

$$y = x^3 + x$$
$$y = (-x^3) + x$$
$$f(x) = -f(-x)$$

$$f(x) = \frac{f(x) + f(-x)}{2} \text{ even}$$
$$f(x) = \frac{f(x) - f(-x)}{2} \text{ odd}$$
Stationary Points:
\[
\frac{df}{dx} = 0 \\
\{ f'(x) = 0 \} \\
x = x_1
\]

Physical Interpretation
\[
\begin{align*}
A &= 160t - 16t^2 \\
\theta_{\text{max}} &= 7 \\
\omega &= 0
\end{align*}
\]
Maxima-Minima

Fundamental Theorem:
Suppose \( f(c) \geq f(x) \) for all \( x \neq c \) and suppose \( f'(c) \) exists. Then \( f'(c) = 0 \).

(a) Beware of False Converse:
\[ f(x) = x^2, \quad \text{dom} f = \mathbb{R} \]
\[ f'(x) = 2x \]
\[ f'(0) = 0 \]
\[ f'(c) \] does not exist.

(b) Beware of Endpoints:
\[ f(x) = x^2, \quad \text{dom} f = [2, 3] \]
\[ f'(x) = 2x \]
\[ f'(0) = 0 \]
\[ f'(c) \] does not exist.

If \( f \) has a max or min, it does not occur in \((0, 1)\).
Graphical Interpretation:

- \( y = x^2 \)
- \( x > 0 \)
- \( x = 0 \)
- \( V = \pi x^2 \)

\( x = 2 \)

\( V = \pi xy \)

- \( A_2 = 45 \text{ cm}^2 \)
- \( V_2 = \pi (15)^2 \text{ cm}^3 \)
- \( A_2 = 45 \text{ cm}^2 \)
- \( V_2 = \pi (15)^2 \text{ cm}^3 \)

\( V = \pi (2y) \)

\( \frac{dV}{dx} = 30\pi y \text{ cm}^3/\text{cm} \)

\( V = \pi x^2 \text{ cm}^3 \)

\( x = 2 \text{ cm} \)

\( x = 0 \text{ cm} \)

\( x = 1 \text{ cm} \)

\( x = 2 \text{ cm} \)

\( V = \pi x^2 \text{ cm}^3 \)

\( x = 2 \text{ cm} \)

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\( x = 2 \text{ cm} \)

\( x = 0 \text{ cm} \)

\( x = 1 \text{ cm} \)

\( V = \pi x^2 \text{ cm}^3 \)
Rolle's Theorem and its Consequences

Let $f$ be defined and continuous on $[a, b]$ and differentiable in $(a, b)$. Suppose also $f(a) = f(b) = 0$. Then $f'(c) = 0$ for at least one $c \in (a, b)$.

Cautions:
1. There may be several $c$.

2. Curve must be smooth.

3. $f$ must be single-valued.
The Mean Value Theorem

Let \( f \) be continuous on \([a, b]\) and differentiable in \((a, b)\). Then there exists a number \( c \in (a, b) \) such that

\[
\frac{f(b) - f(a)}{b - a} = f'(c)
\]

Caution:

Mean Value Theorem Supplies Rigor to Intuition

Examples:

1. \( F(x) = 0 \rightarrow \frac{F(b) - F(a)}{b - a} = F'(c) \)
   - \( F(b) = F(a) = 0 \)
   - \( F'(c) = 0 \)
Inverse Differentiation

\[ f'(x) = D(f(x)) \]

\[ D(x^2) = 2x \]

Suppose \( D(x^2) = 2x \)

\[ f(x) = x \quad \text{constant} \]

\[ \therefore f(x) \in \{ x^2 + c \} \]

In general:

Suppose we are given \( f(x) \)

Let \( E(f) = \{ f(x) + c \} \)

\[ D^{-1}(f(x)) = 2x \]

So:

\[ D'\left(\frac{2x^2}{x^2+1}\right) = \frac{4x(x^2+1)}{(x^2+1)^2} \]

But even if we didn't know this, we would still have

Example: Let \( h(x) = x^2 + \frac{1}{x} \)

\[ h'(x) = 2x - \frac{1}{x^2} \]

\[ h'(x) = \frac{2}{x^2} - 1 \]

\[ h'(x) = \frac{2}{x^2} \]

\[ h'(x) = \frac{1}{x^2} \]
**Chain Rule**

\[
D [f(g(x))] = f'(g(x)) \cdot g'(x)
\]

**Example**

\[
D (x^2 + 3x) = 2x + 3
\]

**Inverse Chain Rule**

\[
\frac{d}{dx} \left( \frac{1}{f(x)} \right) = -\frac{f'(x)}{[f(x)]^2}
\]

**Example**

\[
\frac{d}{dx} \left( \frac{1}{x^2} \right) = -\frac{2}{x^3}
\]

**Inverse Function**

\[
D (f^{-1}(x)) = \frac{1}{f'(f^{-1}(x))}
\]

**Example**

\[
D (f^{-1}(x)) = \frac{1}{f'(f^{-1}(x))}
\]

**Integration**

\[
\int (x^2 + 3x) \, dx = \frac{x^3}{3} + \frac{3x^2}{2} + C
\]

**Integration by Parts**

\[
\int f(x)g'(x) \, dx = f(x)g(x) - \int f'(x)g(x) \, dx
\]

**Example**

\[
\int \sin(x) \, dx = -\cos(x) + C
\]
The "Definite" Indefinite Integral

\[ D(f(x)) = \int f(x) \, dx \]

\( = \{G(x)/G'(x)\} \)

\( = \{F(x)/F'(x)\} \)

\[ f(x) = x^2 \]

\[ \frac{d}{dx} x^2 = 2x \]

\[ G(x) = \frac{x^3}{3} + C \]

Suppose \( f(x) \rightarrow 30 \)

\[ c = \frac{30}{3} \]

\[ f(x) = \frac{3}{2} x^2 + \frac{30}{3} \]

\[ \frac{dy}{dx} = 2x + \frac{10}{3} \]

Notice that if \( H \) is a function of \( x \)

\[ H' = G' \]

\[ H(x) = G(x) + C \]

For this case

\[ H(a) = G(a) + C \]

\[ H(b) = G(b) + C \]

\[ H(a) - H(b) = G(a) - G(b) \]

Geometric Interpretation

\[ f(x) \]

\[ G(x) \]

\[ y = f(x) \]

\[ x = G(x) \]
Physical Interpretation

\[ x(t) = \frac{1}{2} t^2 + \frac{1}{2} \]
\[ v(t) = x'(t) = t \]
\[ a(t) = v'(t) = 1 \]

Generalization

\[ x(t) = \frac{1}{2} t^2 + C \]
\[ v(t) = x'(t) + \frac{dx}{dt} \]
\[ a(t) = v'(t) = \frac{dx}{dt} \]

\[ x(0) = 0, \quad v(0) = \frac{dx}{dt} \]
\[ x(t) = \frac{1}{2} t^2 + \frac{1}{2} \]
\[ \Delta x(t) = \frac{1}{2} t \]

Once we invent \[ \int f(x) \, dx \] to denote \[ \{ a(t), G'(t) \} \]

Why not invent \[ \int f(x) \, dx \]

\[ G(b) - G(a) \]
\[ G(b) = \int_a^b f(x) \, dx \]
\[ G(a) = \int_a^a f(x) \, dx \]
\[ G(b) - G(a) = \int_a^b f(x) \, dx \]

Summary

Suppose \[ \int \frac{dx}{dt} \, dt = F(t), \quad a \leq t \leq b \]

Then \[ \int_a^b \frac{dx}{dt} \, dt = F(b) - F(a) \]

where \( G' = f \)
Block III: The Circular Functions

3.010 Circular Functions

35 min.
\[ f(x) = \sin x \]
\[ f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \]
\[ = \lim_{h \to 0} \frac{\sin(x+h) - \sin x}{h} \]
\[ = \lim_{h \to 0} \frac{\sin x \cos h + \cos x \sin h - \sin x}{h} \]
\[ = \lim_{h \to 0} \frac{\sin x (\cos h - 1) + \cos x \sin h}{h} \]
\[ = \cos x \]
\[ \lim_{t \to 0} \frac{\sin t}{t} = 1 \]

\[ x = \sin kt \]
\[ \frac{dx}{dt} = k \cos kt \]
\[ \frac{dx}{dt} = -k^2 \sin kt \]
\[ d\left(\cos^2 \frac{x}{2}\right) = -2 \cos x \sin x \frac{dx}{dt} \]
\[ \int \sin x \cos x \, dx = -\frac{1}{2} \cos^2 x + C \]

\[ y = \tan x \]
\[ \frac{dy}{dx} = \sec^2 x \]
\[ \int \sec^2 x \, dx = \tan x + C \]

\[ \frac{dy}{dx} = \cos x \]
\[ \int \frac{dx}{\sqrt{1-x^2}} = \sin^{-1} x + C \]

\[
\cos \theta = x \\
\sin \theta = \sqrt{1-x^2} \\
\cos \theta \cdot d\theta = -dx \\
\int \frac{1}{\sqrt{1-x^2}} \cos \theta \cdot d\theta = \sin^{-1} x + C \\
B + C = \sin^{-1} x + C
\]

\[
\cos^2 x = \frac{1}{2} - \sin^2 x \\
\sin^2 x = \frac{1}{2} - \cos^2 x \\
\]

\[
\frac{d}{dx}(\cos^2 x) = -2 \sin x \cos x \\
\frac{d}{dx}(\sin^2 x) = 2 \cos x \sin x \\
\]

\[
\frac{d}{dx}(\frac{1}{\sqrt{1-x^2}}) = -\frac{1}{\sqrt{1-x^2}^3} \\
\]

\[
\frac{d}{dx}(\sqrt{1-x^2}) = -\frac{1}{2\sqrt{1-x^2}} \\
\]
Block IV: The Definite Integral

4.010  2-dimensional Area  36 min.

Method of Exhaustion:
Given a region $R$, we "squeeze it" between two networks of rectangles.

Example:
We wish to determine the area, $A_R$, of the region $R$ where:

\[
\int_{a}^{b} f(x) \, dx
\]

\[
\begin{align*}
\text{Area} & \quad \text{Integral Calculus} \\
& \quad \text{Bible Calculus} \\
\text{Ancient Egyptians} & \quad \text{140 B.C.} \\
\end{align*}
\]

Known Fact:
(i) $A=bh$, for a rectangle
(ii) $R$-sum implies $A \geq A_R$
(iii) $R=R_0 + R_k$, then $A_R = A_0 + A_k$

For each $n$:
\[
L_n = a_k < L_k < U_k
\]
\[
\lim_{n \to \infty} L_n = \lim_{n \to \infty} U_n
\]
\[
L_n = f(a_k) \Delta x_k
\]
\[
U_n = f(b_k) \Delta x_k
\]
\[
L_n = \frac{1}{n} \left( 1^2 + 2^2 + \cdots + n^2 \right)
\]
\[
U_n = \frac{1}{n} \left( 0^2 + 1^2 + \cdots + (n-1)^2 \right)
\]

\[
\begin{align*}
L_{1000} & \approx 0.3827335 \\
U_{1000} & \approx 0.385333 \quad A_R \approx 0.38
\end{align*}
\]
For each $n$, $u_n > \frac{1}{2}$
\[
\frac{1}{u_n} = \frac{1}{1 + \frac{1}{2} (u_n - \frac{1}{2})} = \frac{2}{u_n - \frac{1}{2}}
\]
Similarly, $v_n$

For each $n$, $\frac{1}{v_n} = \frac{2}{v_n - \frac{1}{2}}$

**Generalization**

Let $f$ be continuous on $[a, b]$ (and non-decreasing) and define
\[
A(f) = \int_a^b f(x) \, dx
\]

Partition $[a, b]$ into $n$ equal parts, $x_0, x_1, \ldots, x_n$, and define
\[
A_n = \sum_{i=1}^{n} f(x_i) (x_i - x_{i-1})
\]

Then (1) $A_n$ is a lower sum for $f$
(2) Let $c_i = (x_i - x_{i-1})/2$
\[
\int_a^b f(x) \, dx = \lim_{n \to \infty} A_n = \lim_{n \to \infty} \sum_{i=1}^{n} f(x_i) (x_i - x_{i-1})
\]

**Trapezoidal Approximations**

**Definition**

4 is called piecewise continuous on $[a, b]$ if $f$ is continuous except at a finite number of points where it has 'jump' discontinuities.
4.020  Marriage of Differential & Integral Calculus  30 min.

First Fundamental Theorem of Integral Calculus

Suppose we know exactly a function $G$ such that $G' = f$. Then $A(x) = G(x) + C$

Since $A(a) = 0$, we have:

$$0 = f(a) G(a) + C$$

Let $h = b - a$

Then:

$$A(b) = f(b) G(b) - G(a)$$

But:

$$A(b) = \lim_{h \to 0} \frac{\Delta A}{\Delta x} = A'(x)$$

That is, we can compute

$$\lim_{h \to 0} \frac{2 \Delta(f(x))}{h}$$

to be use of "inverse" derivates.

Example:

$$\lim_{h \to 0} \frac{2 \Delta(f(x))}{h}$$

Historically, $\Delta$ f(x) was "invented" to denote $\ln \frac{x}{0.6}$. Then by the First Fundamental Theorem:

$$\int_0^x f(x) dx = G(x) - G(a), G' = f$$
Second Fundamental Theorem of Integral Calc.

Suppose we want $A_x$,

\[ A_x = \int_1^x f(t) \, dt \]

Then:
\[ G(x) = \int_1^x f(t) \, dt \]

But I don't know yet what $G$ is.

In fact, because we can pinpoint $A_x$, we can construct $G$ such that $G'(x) = \frac{f(x)}{x}$. Namely:

\[ G(x) = \int \frac{f(t)}{t} \, dt \]

The graph shows $G(x)$ and $A(x)$ for $x \geq 1$.

In general:
let $f$ be continuous on $[a,b]$, define $G$ by:

\[ G(x) = \int_a^x f(t) \, dt \]

Then:
\[ G'(x) = f(x) \]

Summary:
(1) First find $A(x)$ allows us to compute $\int_a^b f(x) \, dx$

provided we can find $G$ such that $G' = f$. In this case:
\[ \int_a^b f(x) \, dx = G(b) - G(a) \]

(2) Second find $A(x)$ allows us to give $f$ to construct $G$, such that $G' = f$. Namely:
\[ G(x) = \int_a^x f(t) \, dt \]

\[ A(x) = \int_a^x f(t) \, dt \]

And we may therefore compute $\ln x$ as an indefinite sum.
Cylindrical Shells

\[ V = \pi \int_a^b x f(x) \, dx \]

\[ V = \pi \left[ \frac{1}{2} (2x^2 - 2^2) \right]_0^2 \]

\[ V = \pi \left[ \frac{1}{2} (8x^2 - 4) \right]_0^2 \]

\[ V = \pi \left[ 8x^2 - 4 \right]_0^2 \]

\[ V = \pi \left[ 2 \left( \frac{4}{3} \right) \right] \]

\[ V = \frac{8\pi}{3} \]

\[ \Delta x \approx \frac{1}{n} \]

\[ n \approx \frac{1}{\Delta x} \]

\[ n \approx \frac{2}{\Delta x} \]

\[ f \text{ cont on } [a, b] \]

\[ a = x_0 < x_1 < \ldots < x_n = b \]

\[ Q = \lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} f(x_i) \Delta x \]

\[ Q = \int_a^b f(x) \, dx \]

\[ G(b) - G(a), \quad G = f \]
4.040 1-dimensional Area (Arc Length) 36 min.

1-dimensional Area (arc length)

**Axiom 1**
we can measure the length of any straight line segment.

**Axiom 2**
The length of the whole equals the sum of the lengths of the parts.

**Intuitive Approach**
Lay off a "string" along $AB$.
Then straighten the string and measure its length with a ruler.

**Analytical Approach: Trial #1**

**Analytical Approach: Trial #2**

However, it need not be true that $RcS - lb \le L$.
For example:

The "three questions"
1. Does the limit $L^1$ exist?
2. If so, how do we compute it?
3. How does $L^1$ compare with our intuitive ideas about arc length?
Answer to Question 2:

\[
L_b = \lim_{x \to 0} \frac{\int_0^x f(t) \, dt}{x^2}
\]

where \(f(t) = \frac{1}{\sqrt{1 + t^2}}\).

Now if \(f\) is continuous, so also are \(f(0)\), \(1/\sqrt{1 + t^2}\) and \(\int_0^x f(t) \, dt\). Thus, in this case

\[
L_b = \lim_{x \to 0} \frac{\int_0^x f(t) \, dt}{x^2}
\]

which may be hard to evaluate.

Summarizing:

If \(f\) is differentiable on \([0, b]\) and \(f'\) is continuous on \([0, b]\), then

\[
L_b = \lim_{x \to 0} \frac{\int_0^x f(t) \, dt}{x^2}
\]

Generalization of Question 2:

Suppose \(\phi\) is any function defined on \([a, b]\) and we assume that \(f \circ \phi(x) \neq \phi(x)\) where \(\phi\) is some "little" function defined on \([0, b]\).

Then:

\[
\lim_{x \to 0} \frac{\int_0^x (f \circ \phi(t)) \, dt}{x^2} = 0
\]

and if \(f\) is continuous on \([0, b]\),

\[
\lim_{x \to 0} \frac{\int_0^x (f \circ \phi(t)) \, dt}{x^2}
\]

denoted by \(\phi\) is.

The question is: Does \(\phi\) exist?
Block V: Transcendental Functions

5.010 Logarithms without Exponents  

34 min.

Logarithms Without Exponents

\[ \ln x = \int_{1}^{x} \frac{1}{t} \, dt \]

\[ d \ln x = \frac{dx}{x} \]

\[ \frac{dm}{m} = -\frac{dk}{k} \]

\[ \int \frac{dm}{m} = \ln k + C \]

Problem:
To determine \( L(x) \)
Such that \( L(x) = \frac{1}{x} \)
(This is the case of \( \int x^{-1} \, dx \))

Differential Calculus Approach

\[ y = L(x) \quad \frac{dy}{dx} = \frac{1}{x} \]

Integral Calculus Approach

Pick a \( x_0 \) and define
Picked \( \frac{dx}{dt} \) at.

Define \( L(x) \) by

\[ L(x) = \int_{1}^{x} \frac{dt}{t} \]

\[ L'(x) = \frac{1}{x} \]

\[ y = L(x) \]

\[ x = x_0 \]

\[ y = L(x) \]

\[ h = \ln(2) \]
Logarithmic Functions

If $f$ is called logarithmic

then $f(x, y) = f(x) + f(y)$

for all $x, y$ in dom $f$.

If $f$ is logarithmic,
then:

1. $f(1) = 0$, since $f(1) = f(1) + f(1)$.

2. If $f(x)$ is defined
then $f(x^n) = n f(x)$, since $f(x) = f(x^{n/2} + x^{n/2})$.

Example

With "traditional" logarithms, if the base is $b$ then

$$\log_b b = 1$$

Thus if $e$ is to be the
"base" for $\ln x$, then

$$\ln e = 1$$

Let's compare

$L(bx)$ with $L(b)+L(x)$

$$\frac{dL(bx)}{dx} = \frac{dL(b)+dL(x)}{dx}$$

$$\frac{dL(bx)}{dx} = \frac{1}{bx} \cdot b = \frac{1}{x}$$

Therefore:

$$L(x) = L(1) + L(x) + C$$

If $x = 1$,

$L(1) = L(1) + L(1) + C$

Thus:

$L(1) = 0$
Inverse Logarithms

Claim
\[ \ln^{-1}(x+y_z) = (e^{-z} y_z) (e^{x} - z) \]
\[ y_z = e^{-z} y_z \]
\[ x_z = e^{x} \]
\[ z \]
\[ x_z y_z + z y_z = e^{x} \]
\[ y_z \]
\[ \ln^{-1}(x+y_z) = y_z \]

Find \( \frac{dy}{dx} \) if \( y = e^{-z} x \)
\[ y = e^{-z} x \rightarrow x = z y \]
\[ \frac{dx}{dy} = \frac{1}{z} \]
\[ \frac{dy}{dx} = \frac{1}{\frac{dx}{dy}} \cdot \frac{dx}{dy} \]
\[ \frac{d(e^{-z} x)}{dx} = e^{-z} \frac{dx}{dy} \]
\[ \int \frac{dz}{y} \int dx \]

Notation
\( e_z \) is usually abbreviated \( e_z(x) \)
This matches the identification of \( e_z \) with \( \ln e_z \)
\[ \frac{d(e^z)}{dx} = e^z \]
\[ \frac{d e^x}{dx} = e^x \frac{du}{dx} \]

\[ \int e^{-x^2} dx = \frac{-e^{-x^2}}{2} + C \]

\[ \int 2x e^{-x^2} dx = \frac{-e^{-x^2} + C}{2} \]

\[ \int e^{-x^2} dx = \frac{-e^{-x^2}}{2} + C \]

\[ \int 2x e^{-x^2} dx = \frac{-e^{-x^2}}{2} + C \]

\[ y'' + ay' + by = 0 \]

Try: \[ y = e^{rx} \]

\[ y' = re^{rx} \]

\[ y'' = r^2 e^{rx} \]

\[ r^2 e^{rx} - 5re^{rx} + 6e^{rx} = 0 \]

\[ e^{rx} (r^2 - 5r + 6) = 0 \]

\[ r = 2 \text{ or } r = 3 \]

In general, if \( a \) and \( b \) are constants, \( y = e^{rx} \) transforms into \( y'' + ay' + by = 0 \)
what a Difference
a Sign Makes

\(
\begin{align*}
\{ & x^2 + y^2 = 1 \\
\text{Circular Functions} & \\
\{ & x^2 - y^2 = 1 \\
\text{Hyperbolic Functions} & \\
\end{align*}
\)

Aside
\(\sin x = \frac{a}{x}\) means
\(\cos x = \frac{b}{x}\)
where \(x^2 = 1\)

(a) Given \(a\) and \(b\) let \(a = 1/\sqrt{2}\)

let \(x = a + b\)

\(y = a - b\)

then \(x^2 = 2\)

\(y^2 = 2\)

\(x + y = 2\)

\(x - y = 0\)

\[
\begin{align*}
\frac{\partial (x^2 + y^2)}{\partial x} & = 2x \\
\frac{\partial (x^2 + y^2)}{\partial y} & = 2y \\
\frac{\partial (x^2 - y^2)}{\partial x} & = 2x \\
\frac{\partial (x^2 - y^2)}{\partial y} & = -2y \\
\text{Let} \ C(t) = e^{x t} \\
S(t) = e^{-x t} \\
\Rightarrow \ C'(t) = S(t) \\
S'(t) = -C(t) \\
\end{align*}
\]

\[
\begin{align*}
\cos t & = \frac{C(t) + S(t)}{2} \\
\sin t & = \frac{C(t) - S(t)}{2} \\
x = \cos t \\
y = \sin t \\
& \Rightarrow x^2 + y^2 = 1, x > 0
\end{align*}
\]
\[ y = \tanh x \]
\[ \frac{dy}{dx} = \frac{e^x - e^{-x}}{e^x + e^{-x}} \]
\[ \frac{d}{dx}(\tanh x) = \text{sech}^2 x \]
\[ \int \text{sech}^2 x \, dx = \tanh x + C \]

\[ y = \text{sech} x = \frac{e^x + e^{-x}}{2} \]
\[ y' = \text{sech} x = \frac{e^x - e^{-x}}{2} \]
\[ y'' = \text{sech} x = \frac{e^x + e^{-x}}{2} \]

\[ \int \text{sech}^2 x \, dx = \tanh x + C \]

Hyperbolic Functions are a solution to
\[ \frac{d^2 x}{dt^2} + K x = 0 \]

Circular Functions are a solution to
\[ \frac{d^2 x}{dt^2} - K x = 0 \]
Inverse Hyperbolic Functions

\[ y = \sinh^{-1} x \]
\[ x = \sinh y \]
\[ \frac{dy}{dx} = \cosh y \]
\[ \frac{dx}{dy} = \cosh y \]
\[ \cosh^2 y - \sinh^2 y = 1 \]
\[ \cosh y = \sqrt{1 + x^2} \]
\[ \frac{d}{dx} (\sinh^{-1} x) = \frac{1}{\sqrt{1 + x^2}} \]

\[ \int \frac{dx}{\sqrt{1 + x^2}} = \sinh^{-1} x + C \]

\[ \tan \theta = 2 \]
\[ \sec^2 \theta \, d\theta = dx \]
\[ \frac{d}{dx} \sqrt{1 + x^2} = \sec \theta \]
\[ \frac{d}{dx} \tan \theta = \sec^2 \theta \]
\[ \frac{1}{2} \tan \theta = \text{Area} \]
\[ \sec \theta = \sqrt{1 + x^2} \]
\[ \tan^{-1} \theta = \text{Area} \]
\[ \sec^{-1} \theta = \text{Area} \]
\[ \sinh^{-1} x \]
\[ y = \sinh^2 x \frac{\ln(x + \sqrt{x^2 + 1})}{2} \]

\[ x = \sinh y \]
\[ e^x - e^{-x} = 2 \]
\[ e^y = -e^{-y} \]
\[ c^2 - 2x - 1 = 0 \]
\[ c^2 - 2x + 2(c^2) + 1 = 0 \]
\[ c^2 = 2x \pm \sqrt{3x^2 + 4} \]
\[ c^2 = x \pm \sqrt{x^2 + 1} \]
\[ y = \ln(x + \sqrt{x^2 + 1}) \]

\[ \text{Define} \]
\[ C_1(x) = \cosh x, \ x > 0 \]
\[ C_2(x) = \cosh x, \ x < 0 \]

Then \( C_1 \) and \( C_2 \) are each 1-1.

\[ C_1', C_2' \text{ exist} \]

\[ \frac{d}{dx} \cosh x = \sinh x \]
\[ \frac{d}{dx} \sinh x = \cosh x \]

\[ \cosh^2 x = C_1(1) \]

That is:

\[ y = \cosh x \]
\[ x = \cosh y, \ y > 0 \]
\[ \frac{dx}{dy} = \sinh y, \ y > 0 \]
\[ \frac{dy}{dx} = \cosh y \]
\[ \cosh^2 y - \sinh^2 y = 1 \]

\[ \sinh y = \frac{\sqrt{x^2 - 1}}{x} \]
\[ 2y = \sinh^{-1} x \]
\[ \frac{d}{dx} \cosh^{-1} x = \frac{1}{\sqrt{x^2 - 1}} \]
\[ \frac{d}{dx} \sinh^{-1} x = \frac{1}{\sqrt{x^2 - 1}} \]
Block VI: More Integration Techniques

6.010 Some Basic Recipes

30 min.

**Some Basic Recipes**

Pick a differentiable function $G$, say $G = f$.
Then:
$$\int f(x) \, dx = G(x) + C$$

Given $f$, the required $G$ may not exist in 'familiar' form.
For example:
$$\int e^x \, dx = e^x$$

Objective of this block is to find 'recipes' for finding $G(x)$ for various types of $f(x)$.

\[
\begin{align*}
\int u^n \, du & = \frac{u^{n+1}}{n+1} + C, \quad n \neq -1 \\
\int \cos^m x \cos x \, dx & = \frac{\cos^{m-1} x \sin x}{m} + C, \quad m \neq 1 \\
\int \sin^m x \cos^n x \, dx & = -\frac{\sin^{m-1} x \cos^{n+1} x}{m+n} + C, \quad m, n \neq -1 \\
\int \sin x \cos x \, dx & = \frac{\sin^2 x}{2} + C \\
\int \cos^2 x \, dx & = \frac{x}{2} + \frac{\sin 2x}{4} + C \\
\int \sin^2 x \, dx & = -\frac{x}{2} + \frac{\cos 2x}{4} + C \\
\end{align*}
\]
Sums and Differences of Squares

\[ \int \frac{dx}{\sqrt{a^2 - x^2}} = \arcsin \left( \frac{x}{a} \right) + C \]

Completing the Square

\[ a \left( x^2 + \frac{b}{a} x + c \right) = a \left( x^2 + 2 \cdot \frac{b}{2a} x + \left( \frac{b}{2a} \right)^2 \right) - a \left( \frac{b}{2a} \right)^2 \]

\[ \int \frac{dx}{\sqrt{x^2 + \frac{b}{a} x + c}} = \frac{1}{a} \int \frac{du}{\sqrt{u^2 + \frac{b}{2a} u + \left( \frac{b}{2a} \right)^2}} \]

Let \( u = x + \frac{b}{2a} \)
Partial Fractions

Technique applies to \( \int \frac{P(x)}{Q(x)} \, dx \) where \( P \) and \( Q \) are polynomials in \( x \) (deg \( P \leq \deg Q \)).

We can’t factor \( x + 1 \) without using non-real numbers.

\[
\frac{x^2 + 1}{(x + 1)^2} = \frac{(x + 1)(x - 1)}{(x + 1)^2} - \frac{x}{x + 1}
\]

\[
\frac{x^2 + 1}{x + 1} = \frac{x}{x + 1} - \frac{1}{x + 1}
\]

We can handle linear and quadratic denominators.

(2) Theoretically, every real polynomial can be factored into linear and quadratic terms.

\[
\int \frac{dx}{(x-1)(x-2)^2} = \frac{1}{2} \ln|\frac{x-1}{x-2}| - \frac{1}{2} \ln|x-2| - \frac{1}{4} \int \frac{dx}{x-2}
\]

\[
\int \frac{dx}{(x+1)^2} = \frac{1}{2} \int \frac{dx}{x+1} - \frac{1}{2} \ln|x+1| + C
\]
Polynomial Identities

\[
\begin{align*}
& \frac{a^2}{2} + \frac{b^2}{2} + \frac{c^2}{2} + \frac{d^2}{2} \\
\text{(i)} & \quad \text{Let } x = a, y = b, \\
& \quad \text{then } (a, b) = x(b, y) \\
& \quad = ax^2 + bx + c \\
\text{(ii)} & \quad \text{Let } x = a, y = b, \text{ etc.} \\
& \quad 2ax + 2b = 2bx + 2c \\
& \quad 2a + 2b = 2b + 2c
\end{align*}
\]

Beware:

In general

\[
3^x + 5^x = 3y + 5^x
\]

does not imply that

\[
3^x = 3y \quad \text{and} \quad 3^x = 5^x
\]

Example:

\[
\frac{5}{x} \quad \frac{5}{x} \quad \frac{5}{x}
\]

But

\[
6 \neq 10
\]
Integration by Parts

\[ \int c \, dx = cx + C \]

Example:

Let \( u = x \) and \( v = \cos x \)

Then

\[ \int x \cos x \, dx = \int x \, d(\cos x) \]

\[ = x \cos x - \int \cos x \, dx \]

\[ = x \cos x - \sin x + C \]

Check: \( x \cos x + \sin x - x \cos x = x \cos x \)

\[ \int \frac{dx}{x} = \ln |x| + C \]

\[ \int \frac{dx}{x^2} = -\frac{1}{x} + C \]

\[ \int x^2 \cos x \, dx = \int x^2 \, d(\sin x) \]

\[ = x^2 \sin x - \int 2x \sin x \, dx \]

\[ = x^2 \sin x + 2 \int x \sin x \, dx \]

\[ = x^2 \sin x + 2(\cos x - x \sin x) + C \]

\[ \int x^2 \cos x \, dx = \int x^2 \, d(\sin x) \]

\[ = x^2 \cos x - \int 2x \cos x \, dx \]

\[ = x^2 \cos x + 2 \int x \cos x \, dx \]

\[ = x^2 \cos x + 2(\sin x - x \cos x) + C \]

\[ \int x \cos x \, dx = \int x \, d(\sin x) \]

\[ = x \sin x - \int \sin x \, dx \]

\[ = x \sin x + \cos x + C \]

\[ \int \sin x \, dx = -\cos x + C \]

\[ \int \cos x \, dx = \sin x + C \]

\[ \int \sec x \, dx = \ln |\sec x + \tan x| + C \]

\[ \int \csc x \, dx = -\ln |\csc x + \cot x| + C \]

\[ \int \tan x \, dx = -\ln |\cos x| + C \]

\[ \int \cot x \, dx = \ln |\sin x| + C \]

\[ \int \sec^2 x \, dx = \tan x + C \]

\[ \int \csc^2 x \, dx = -\cot x + C \]
\[ \int \frac{\tan^2 x}{x} \, dx \]
\[ u = \tan^2 x \quad dv = dx \]
\[ du = \frac{dx}{1 + \tan^2 x} \quad v = x \]
\[ \int \tan^2 x \, dx = x \tan^2 x - \int \frac{x \tan^2 x}{1 + \tan^2 x} \, dx \]
\[ = \frac{x}{2} \tan^2 x - \frac{1}{2} \ln(\sec^2 x + 1) + C \]

\[ \int \ln x \, dx \]
\[ u = \ln x \quad dv = dx \]
\[ du = \frac{1}{x} \, dx \quad v = x \]
\[ \int \ln x \, dx = x \ln x - x + C \]
\[ k_x = \int_{a}^{b} x \, dx = \frac{1}{2} b^2 - \frac{1}{2} a^2 + C \]
Improper Integrals

Find the flaw:

\[ \int_{\frac{1}{2}}^{\infty} \frac{1}{x^2} \, dx = G(1) - G(\frac{1}{2}) \]

\[ G(1) = \infty, \quad G(\frac{1}{2}) = \frac{1}{2} \]

\[ \int_{\frac{1}{2}}^{\infty} \frac{1}{x^2} \, dx = \frac{1}{2} - 0 \]

\[ (\frac{1}{2} \geq 0, \quad \int_{\frac{1}{2}}^{\infty} \frac{1}{x^2} \, dx > 0) \]

key Point:

\[ \int_{a}^{b} f(x) \, dx = G(b) - G(a) \]

\[ G'(x) = f(x) \]

requires that \( f \) be

(piecewise) continuous on \([a,b]\)

\[ \frac{1}{2} = \frac{1}{2} \quad \text{when} \quad x = 0, 0 \in (a,b) \]

Definition #1:

\[ \int_{a}^{b} f(x) \, dx \]

is called improper

of the first kind \( \leftrightarrow \) \( f \) is

infinite for at least one \( x \in [a,b] \)

If \( c \) is the only point

in \((a,b)\) at which \( f \) is infinite

we define

\[ \int_{a}^{b} f(x) \, dx = \lim_{h \to 0} \left[ \int_{a}^{c-h} f(x) \, dx + \int_{c+h}^{b} f(x) \, dx \right] \]

Pictorially:

\[ \text{Area of an infinite region} \]

\[ \int_{a}^{b} f(x) \, dx \]

The question centers

about whether \( \lim_{h \to 0} \)

"poses a time" \( f(x) \) is

In our example this

didn't happen.

\[ \text{Im} \left[ \int_{a}^{b} f(x) \, dx \right] = \text{Im} \left[ \lim_{h \to 0} \int_{a}^{c-h} f(x) \, dx + \int_{c+h}^{b} f(x) \, dx \right] \]

\[ = \infty - 0 \]
On the other hand, consider

\[ \int_{\frac{1}{\sqrt{2}}}^{\frac{1}{2}} x^2 \, dx = \frac{1}{4} - \frac{1}{8} = 0.125 \]

\[ \int_{\frac{1}{2}}^{\frac{1}{\sqrt{2}}} x^2 \, dx = \frac{1}{4} - \frac{1}{8} = 0.125 \]

In both cases, the integral converges.

Definition #2

If \( f \) is infinite at \( x = a \), \( c \in (a, b) \) and the improper integral \( \int_{a}^{c} f(x) \, dx \) is convergent

\[
\lim_{x \to c^-} \int_{a}^{x} f(t) \, dt = F(c) - F(a)
\]

otherwise it is called divergent.

Examples:

1. \( f(x) = \frac{1}{x^2} \) and \( f(x) = \frac{1}{\sqrt{x}} \) are inverses.

\[ \int_{1}^{2} x^2 \, dx = \lim_{b \to 0^+} \left[ \frac{x^3}{3} \right]_{1}^{2} = \frac{8}{3} - \frac{1}{3} = \frac{7}{3} \]

Definition #3

\[ \int_{1}^{2} \frac{1}{x^2} \, dx = \lim_{b \to 0^+} \left[ -\frac{1}{x} \right]_{1}^{2} = -\frac{1}{2} + 1 = \frac{1}{2} \]

where \( f(x) \) is continuous for \( x > 0 \). It is called improper of the second kind.

Computational Aside

We do not have to be able to compute \( \int_{c}^{d} f(x) \, dx \) to determine its convergence. For example, consider

\[ \int_{1}^{\infty} \frac{1}{x^2} \, dx \]

\[ \lim_{x \to \infty} \left[ -\frac{1}{x} \right] = 0 \]

\[ \lim_{x \to 1} \left[ -\frac{1}{x} \right] = -1 \]

Infinite Area But Finite Value.

Summary:

The limits of \( f \)

\[ \int_{a}^{b} f(x) \, dx \]

give us warning to beware.

However, always examine \( \int_{a}^{b} f(x) \, dx \) for "infinities" of \( f \).
Block VII: Infinite Series

7.010 Many Versus Infinite

26 min.

---

**Many Versus Infinite**

\[
\begin{align*}
N_{10} &= 10,000,000,000 \\
N_{10} &= 10
\end{align*}
\]

\[N + 1, k + 2, k + 3, \ldots\]

It is no more the end of the number system than \( \infty \).

---

**Additional Examples**

1, 3, 2, 5, 7, 4, 9, 1, 6, \ldots

No matter where you stop (even at \( 10^6 \)) there are twice as many "odds" as "evens".

---

\[
\begin{align*}
1 + 1 + 1 + \cdots &= 0 \quad \left[\text{finite sum}\right] \\
1 + 2 + 3 + \cdots &= 1 \quad \left[\text{infinite sum}\right]
\end{align*}
\]

Notice the need for order as well as the terms.

Why deal with infinite sums? Because we need them.

\[
A_k = \lim_{n \to \infty} \sum_{i=1}^{n} f(x)
\]

How shall we add infinitely many terms?

Consider \( 9 \), \( \frac{1}{2} \), \( \frac{1}{4} \), \( \frac{1}{8} \), \( \frac{1}{16} \), \( \frac{1}{32} \) and we want

\[
\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \ldots
\]

---

Note: Our intuition is defined because it doesn't apply.
Aₙ = ½
A₂ = ½ + ½ = 1
A₃ = ½ + ½ + ½ = 3/2
(Do not confuse the a's and b's. The sequence of numbers being added is 1, ½, ¾, ...)
(5, 5, 5) which the partial
sums in the sequence 0, 1/2, 3/4...
(½, ½, ½). The sum in the
number Aₙ = ½ + ½ + ½ = 3/2

Generalized, if Aₙ = ½
then aₙ = Aₙ = Aₙ - Aₙ₋₁
= Aₙ - Aₙ₋₁ = 2/ₙ
= 1 - 1/n

For example:
A₃ = 1 + 1/3 + 1/5 + 1/7 + ... = 2 - 1/n
= 1 - 1/n

So, how about
Aₙ = 1 - 1/n

Evidently?

As n → ∞, define L = lim Aₙ

Example:

Notice some limit
theorems as before
apply.

Example:

\[ \lim \left( \frac{2n+3}{3n^2} \right) = \lim \left[ \frac{2/n + 3/n^2}{3} \right] \]

That is:
\[ \frac{2}{n^2} \text{ converges to } \lim_{n \to \infty} \frac{2}{n^2} = 0 \]

On the other hand, lim aₙ = 0
so not enough to guarantee
the convergence of \( \sum aₙ \).

Example:

Pictorially:

Basic Definition:
The infinite sequence
b₀, b₁, b₂, ... st.(i) is said
to converge to the limit L
(written lim bₙ = L) if for every number ε > 0, we

\[ \text{can find } N₀ \text{ such that } n > N₀ \Rightarrow \left| bₙ - L \right| < ε \]
Positive Series

Ordering:
\[ S = \{1, 2, 3, 4, 5\} \]
\[ S = \{7, 9, 11, 13\} \]
7 is a lower bound for S
11 is an upper bound for S
7 is the greatest lower bound for S
11 is the least upper bound for S

These results are more subtle for infinite sets:

Examples:
1. Let \( S = \{a, a, a, a, \ldots\} \)
   \( S = \{b, b, b, b, \ldots\} \)
   \( a \neq b \) for \( S \), but \( 1 \neq S \)
2. Let \( S = \{a, a, a, a, \ldots\} \)
   \( a = \{b, b, b, b, \ldots\} \)
   \( 0 \in S \) for \( S \), but \( 0 \notin S \)

Formal Definitions:

M is called an upper bound for \( S \) if \( m > S \) for all \( x \in S \)

A sequence \( \{a_n\} \) is called monotone non-decreasing if
\( a_n \leq a_{n+1} \) for all \( n \)
(That is: \( a, a, a, a, \ldots \))

For such sequences, two possibilities exist:
1. \( \{a_n\} \) has no upper bound
   In this case we write
   \( \lim a_n = \infty \)
   (An example is: \( 1, 2, 3, \ldots \) when \( a = 0 \))

A set is bounded if it has both an upper and a lower bound

Key Property:
Every bounded set has a glb and lub
Pos.ve Series

If $a_n > 0$ for each $n,$ then
$\Sigma a_n$ is called a positive series.

In this case the sequence of partial sums is monotonic non-decreasing.

Therefore $\Sigma a_n$ is a positive series, it either diverges to $\infty$ or it converges to the limit $L$ where $L$ is the lub in the sequence of partial sums.

Comparison Test

Suppose $\Sigma a_n$ is a convergent positive series and $0 \leq b_n \leq a_n$ for each $n.$

Then $\Sigma b_n$ also converges.

Proof

Let $T_{a_n} = \sum_{k=1}^{n} a_k$ and $T_{b_n} = \sum_{k=1}^{n} b_k$ for each $n.$

$\lim_{n \to \infty} T_{b_n} = \lim_{n \to \infty} T_{a_n}$

$b_n \leq a_n$ for each $n.$

$b_n$ is bounded (and monotonic non-decreasing), so $b_n \to \ell$ exists.

Notes

(1) The condition $0 \leq b_n \leq a_n$ for all $n$ can be weakened to $0 < b_n < a_n$ for all $n.$ (i.e., for a "Sufficiently Large") since convergence depends on the "end" of the sequence $1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots - \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \cdots + \frac{1}{n} + \cdots$

(2) If $a_n$ is a positive divergent series, then $\Sigma b_n$ also diverges. Since $a_n$ convergence would imply $b_n$ converges.

Integral Test

Series and Improper Integrals

Suppose there is a decreasing continuous function $f(x)$ such that $f(n) = a_n$ is the $n$th term of the positive series $\Sigma a_n.$

Then $\sum_{n=1}^{\infty} a_n$ converges if and only if $\int_{1}^{\infty} f(x) \, dx$ converges.

Proof

$\int_{1}^{n} f(x) \, dx \leq \sum_{i=1}^{n} a_i \leq a_1 + \int_{1}^{n} f(x) \, dx.$
Absolute Convergence

Consider:
\[ 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n} \]

We see:
1. Terms alternate in sign
2. Term tends to 0
3. Terms decrease in magnitude

Claim: \[ \sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n} \text{ converges} \]

Geometric Proof
\[ \begin{array}{c}
\frac{1}{2} \times \frac{1}{2} < \frac{1}{2} \\
\frac{1}{4} \times \frac{1}{4} < \frac{1}{4} \\
\vdots
\end{array} \]

(\text{It turns out that} \ L = \frac{1}{2}, \ \text{but who'd have guessed it?})

\[ \text{SO WHAT?} \ldots \]

Definition
1. \( \sum_{n=1}^{\infty} a_n \) is said to converge absolutely if \( \sum_{n=1}^{\infty} |a_n| \) converges.

2. A series which converges but not absolutely is called conditionally convergent.

\[ \sum_{n=1}^{\infty} a_n \] is conditionally convergent.
The Subtlety of Conditional Convergence

The sum of a cond. conv. series depends on the order of the terms.

Example:
Divide the terms of \( \sum_{n=1}^{\infty} \frac{1}{n^2} \) into 2 groups (set B):
- \( \mathcal{P} = \{ \frac{1}{1}, \frac{1}{4}, \frac{1}{9}, \frac{1}{16}, \cdots \} \)
- \( \mathcal{N} = \{ \frac{1}{2}, \frac{1}{3}, \frac{1}{5}, \frac{1}{6}, \cdots \} \)

Both \( \sum_{n=1}^{\infty} \frac{1}{n^2} \) and \( \sum_{n=1}^{\infty} \frac{1}{n} \) diverge.

Let us, for example, make the sum \( S_n \) as in the last section and write \( 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} \) until the sum first exceeds \( \frac{\pi}{2} \).

This must happen since the sum is increasing (or as \( n \to \infty \)).

In particular:
- \( 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} + \frac{1}{9} + \frac{1}{10} + \frac{1}{11} + \frac{1}{12} \) We next "cancel" members of \( \mathcal{N} \) until the sum falls below \( \frac{\pi}{2} \).

Summary So Far:

- If \( \sum_{n=1}^{\infty} a_n \) is cond. conv., its limit exists, but the limit changes as the order of the terms is changed. That is, rearranging the terms changes the series.
- This is not true in "fark" arithmetic.
- Don't "monkey" with conditional convergence.

The beauty of absolute convergence is that the sum is the same for every rearrangement of the terms.

Details are left to the supplementary notes.

The "monkey" term: 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} + \frac{1}{9} + \frac{1}{10} + \frac{1}{11} + \frac{1}{12} + \cdots

(\frac{\pi}{2} was not important - though the arithmetic gets messier if we choose a larger number.)

Continue with \( \mathcal{P} \):
- \( 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} + \frac{1}{9} + \frac{1}{10} + \frac{1}{11} + \frac{1}{12} + \cdots \)
- \( 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} + \frac{1}{9} + \frac{1}{10} + \frac{1}{11} + \frac{1}{12} + \cdots \)
- \( 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} + \frac{1}{9} + \frac{1}{10} + \frac{1}{11} + \frac{1}{12} + \cdots \)
- \( 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} + \frac{1}{9} + \frac{1}{10} + \frac{1}{11} + \frac{1}{12} + \cdots \)
- \( 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} + \frac{1}{9} + \frac{1}{10} + \frac{1}{11} + \frac{1}{12} + \cdots \)
- \( 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} + \frac{1}{9} + \frac{1}{10} + \frac{1}{11} + \frac{1}{12} + \cdots \)
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- \( 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} + \frac{1}{9} + \frac{1}{10} + \frac{1}{11} + \frac{1}{12} + \cdots \)
- \( 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} + \frac{1}{9} + \frac{1}{10} + \frac{1}{11} + \frac{1}{12} + \cdots \)}
7.040  Polynomial Approximations

**Polynomial Approximations**

Let \( P_n(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0 \)

If \( P_n(x) = x^n \),

\[ P_n^{(k)}(x) = \binom{n}{k} x^{n-k} \]

**Example:**

Let \( f(x) = e^x \)

Then \( f^{(k)}(x) = e^x \)

\[ P_k(x) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!} (x-a)^k \]

We would like to be able to compare \( f(x) \) with \( \lim_{n \to \infty} P_n(x) \)

This involves 3 questions:

1. Does \( \lim_{n \to \infty} P_n(x) \) exist? \( P(x) = ? \)
2. If so, does it equal \( f(x) \)?
3. Letting \( P(x) = \lim_{n \to \infty} P_n(x) \), does \( P \) possess the polynomial-

property possessed by \( f(x) \)?

**Diagram:**

- \( P_0(x) = 1 \)
- \( P_1(x) = 1 + x \)
- \( P_2(x) = 1 + x + \frac{x^2}{2} \)
- \( P_3(x) = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} \)
Counter-Example

2. Let \( f(x) = \begin{cases} x^2, & \text{if } x \geq 2 \\ 3, & \text{if } x < 2 \end{cases} \)
   \( f(0) = 0, f'(0) = 3, f''(0) = 3 \)
   \( f^{(n)}(0) = 0, n > 2 \)
   \( f^{(n)}(x) = \begin{cases} 2^nx^n & \text{for } 0 < x < 2 \\ 2^nx^n & \text{for } x \geq 2 \end{cases} \)
   \( f^{(n)}(1) = 2^n 
eq f(1) \)

Pictorially

The Ratio Test and Absolute Convergence:

Let \( a_n \) be a sequence of real numbers.

Then \( \sum a_n \) converges if \( \frac{|a_{n+1}|}{|a_n|} \) is bounded by \( L < 1 \).

But we may test \( \sum |a_n| \) for convergence by the ratio test and the root test.

Examples:

1. \( 2^n \) converges \( \Rightarrow |x| < 1 \)
2. \( \sqrt{n} \) diverges \( \Rightarrow |x| > 1 \)
3. \( \sqrt[n]{n} \) diverges (absolutely) \( \Rightarrow |x| > 1 \)
4. \( \sqrt[n]{n} \) converges (absolutely) \( \Rightarrow |x| > 1 \)

"Taylor's Theorem with Remainder"

Using integration by parts repeatedly (as shown in our text), it follows that if \( f \) and its first \( n \) derivatives exist at \( x = a \), then

\[
 f(x) = f(a) + \frac{f'(a)(x-a)}{1!} + \frac{f''(a)(x-a)^2}{2!} + \cdots + \frac{f^{(n-1)}(a)(x-a)^{n-1}}{(n-1)!} + R_n(x)
\]

where

\[
 R_n(x) = \int_a^x \frac{(x-t)^{n-1}f^{(n)}(t)}{(n-1)!} dt
\]
Uniform Convergence

Review: \[ \lim_{n \to \infty} f_n(x) = f(x) \]
for all \( x \in [a, b] \) we say \( \{f_n\} \) converges pointwise to \( f \) on \([a, b] \).

Example: Suppose \( f_n(x) = \frac{x^n}{n} \).
Then \( \lim_{n \to \infty} f_n(x) = 0 \).
Hence \( \{f_n\} \) converges to \( 0 \).

Now \( \lim_{n \to \infty} f_n(0) = 0 \) \( \lim_{n \to \infty} f_n(1) = \frac{1}{n} \to 0 \).
So \( \lim_{n \to \infty} f_n(1) = 0 \).
Note \( \epsilon > 0 \) has to be small.

Of course, \( N \) and \( N_0 \) need not be the same.

Two Basic Definitions
(1) Let \( \lim_{n \to \infty} f_n(x) = f(x) \) on \([a, b] \).
Then \( \lim_{n \to \infty} f_n(x) = f(x) \) for each \( x \in [a, b] \).
That is, \( \lim_{n \to \infty} f_n(x) = f(x) \) for each \( x \in [a, b] \).

Example: \( f_n(x) = \frac{x^n}{n} \).
Then \( \lim_{n \to \infty} f_n(x) = 0 \).
Hence \( \{f_n\} \) converges to \( 0 \).

In general, the choice of \( N \)
depends on the choice of \( x \), and
there are infinitely many such
choices \( (x, N) \).

(2) If we can find one \( N \)
such that \( \lim_{n \to \infty} f_n(x) = f(x) \)
everywhere \( x \in [a, b] \), we say that
the convergence is uniform.

A Pictorial Interpretation

From our picture it should seem clear that \( \{f_n\} \) converges uniformly on \([a, b] \).
and \( f \) is continuous on \([a, b] \).
Then \( f \) is also continuous on \([a, b] \).

(Note: This result, as we have already seen, need not be true for pointwise convergence. Recall our example in which \( f_n(x) = n^2 x \) for \( x \in (0, 1] \).

Since \( f \) is continuous at \( x_0 \), \( f_n \) means
\( \lim_{n \to \infty} f_n(x_0) = f(x_0) \) and since \( f(\bar{x}) = \lim_{n \to \infty} f_n(\bar{x}) \)
for every \( \bar{x} \) in \([a, b] \), we may write \( f(x) = \lim_{n \to \infty} f_n(x) \).
These two results are proven as theorems in the supplementary notes. A theorem about differentiation is also proved.

Differentiation is in a way more subtle than integration since it requires "smoothness" as well as "unbrokenness".

**Example:** Define $f$ on $[0,a]$ by:

$$f(x) = \begin{cases} 
0 & \text{if } x = 0 \\
1 & \text{if } x > 0 
\end{cases}$$

Then $f(x)$ is not continuous at $x = 0$, hence the convergence is not uniform.

\[
\lim_{n \to \infty} \int_0^a f_n(x) \, dx = \int_0^a f(x) \, dx
\]

\[
\lim_{n \to \infty} f_n(x) = 0
\]

\[
\int_0^a f_n(x) \, dx \to \int_0^a f(x) \, dx
\]

Next time we shall show that within the interval of absolute convergence, \( \sum \frac{a_n}{n^2} \) converges uniformly to \( \frac{\pi^2}{6} \).

In other words, within the radius of convergence, \( \frac{1}{x^2} \) "enjoys the usual" polynomial properties associated with \( \frac{1}{x} \).

**Application of Uniform Convergence to Series**

Recall that \( \sum a_n x^n \) is an abbreviation for

\[
\lim_{n \to \infty} \sum_{n=0}^{\infty} a_n x^n
\]

that is, \( a_0 + a_0 x + a_1 x + a_2 x^2 + \cdots \) represents the limit of the sequence:

\[
a_0, a_0 + a_1 x, a_0 + a_1 x + a_2 x^2 + a_3 x^3, \ldots
\]
Uniform Convergence of Series

Weierstrass M-Test

Suppose \( \sum_{n=1}^{\infty} M_n \) is a positive convergent series and that \( |f_n(x)| \leq M_n \) for each \( n \) and each \( x \in [a,b] \).

Then \( \sum_{n=1}^{\infty} f_n(x) \) converges uniformly to \( f(x) := \sum_{n=1}^{\infty} f_n(x) \) on \([a,b]\).

Proof

\[
|f(x) - \sum_{k=1}^{n} f_k(x)| = \left| \sum_{k=n+1}^{\infty} f_k(x) \right| \\
\leq \sum_{k=n+1}^{\infty} |f_k(x)| \leq \sum_{k=n+1}^{\infty} M_k \\
\leq \frac{\sum_{n=1}^{\infty} M_n}{n} \\
\leq \frac{\epsilon}{n} \quad \text{for sufficiently large } n.
\]

An Example:

Let us compute \( \sum_{n=1}^{\infty} \frac{\cos x + \cos 2x + \cdots + \cos nx}{n} \).

By uniform convergence, we have:

\[
\sum_{n=1}^{\infty} \frac{\cos x + \cos 2x + \cdots + \cos nx}{n} = \int \frac{\cos x + \cos 2x + \cdots + \cos nx}{n} \, dx.
\]

Thus,

\[
\sum_{n=1}^{\infty} \frac{\cos x + \cos 2x + \cdots + \cos nx}{n} = \int \left( \frac{\cos x + \cos 2x + \cdots + \cos nx}{n} \right) \, dx.
\]

For each \( x \), \( \frac{\cos x + \cos 2x + \cdots + \cos nx}{n} \) converges uniformly to \( \frac{\sin x + 2 \sin 2x + \cdots + n \sin nx}{n} \) as \( n \to \infty \).
Application to Power Series

Let \( |x| < R \) and
\[
\sum_{n=0}^{\infty} a_n x^n \text{ converge for } n < R
\]
\[
\lim_{n \to \infty} a_n x^n = 0 \left( \text{since } \sum_{n=0}^{\infty} a_n x^n \text{ converges} \right)
\]
Given \( R > 0 \), there exists \( N \) such that \( n > N \) implies \( \left| a_n x^n \right| < \frac{1}{n} \)

Key idea is:
\[
\left| a_n x^n \right| < \left| \frac{(a_0 x^0)^n}{n} \right|
\]
\[
= 10 \cdot x^n \left| \frac{1}{n} \right|
\]
\[
\leq M \left| \frac{1}{n} \right|
\]
Now \( \frac{1}{n} \) is a positive constant < 1
\[
\sum_{n=0}^{\infty} \frac{1}{n} \text{ is a positive convergent (geometric) series}
\]
By Weierstrass M-test,
\[
\sum_{n=0}^{\infty} a_n x^n \text{ is uniformly convergent}
\]
if \( R \) when \( R = \) the radius of convergence for \( \sum_{n=0}^{\infty} a_n x^n \)

Example

Find the area of \( R \)

where:

\[
P_k = \int_0^1 e^{-x^2} \, dx, \text{ but we do not know (explicitly) } g(x)
\]

such that \( g'(x) = e^{-x^2} \)

\[
\sum_{n=0}^{\infty} \frac{1}{n!} \left( \sum_{m=0}^{n} \frac{(-1)^m x^m}{m!} \right)
\]

In other words
\[
A = 1 - \frac{1}{3} + \frac{1}{10} - \frac{1}{45} + \frac{1}{180} + \ldots
\]
\[
\leq 0.749
\]
Resource: Calculus Revisited
Herbert Gross

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