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PROFESSOR:

Hi. In the last unit, we introduced the notion of the structure of vectors in terms of arrows in the plane. And today, we want to talk about the added geometric property of what I could call three-dimensional vectors or three-dimensional arrows.

Now before I say that-- I guess I've already said it-- but before I say any more about it, let me point out, of course, that this is a matter of semantics. Obviously an arrow, being a line, has only one dimension, be it in the direction of the x-axis, be it in the x-y plane, or be it in three-dimensional space. When we say "three-dimensional vectors," we do not negate the fact that we're talking about arrows, but rather that we are dealing with a coordinate system which takes into consideration the fact that the line is drawn through three-dimensional space.

In other words, where as I call the lecture three-dimensional vectors or arrows-- and by the way, both in the notes and as I'm lecturing, I will very often, whenever I write the word "vector," put "arrow" in parentheses, whenever I write the word "arrow," put "vector" in parentheses so that you can see the juxtaposition between these two, the geometry versus the physical or mathematical concept. But the idea is something like this.

In three-dimensional space, we have a natural extension of Cartesian coordinates. Rather than talk about the x-y plane, we pick a third axis, a third number line, which goes through the origin, perpendicular to the x-y plane, and which has the sense that if the x-axis is rotated into the y-axis through the positive 90-degree angle here, that the z-axis is in the direction in which a right-handed screw would turn. I won't belabor this. It's done very, very nicely in the textbook. And I'm sure that you have seen this type of coordinate system before. If you haven't, it's something that takes a matter of some 20 or 30 seconds to pick up.

But at any rate, what we're saying is let's imagine this three-dimensional coordinate system, three-dimensional Cartesian coordinates. The convention is that just as in the plane, we label the point by its x and y components. In three space, a point is labeled by its x , y , and z components. So for example, if I take a vector in three space-- meaning what? It's a line that goes through three dimensional space. I again shift it parallel to itself, so it begins at my origin. And at the risk of causing some confusion here, I still think it's worth doing. Let me call the vector A , and let me call the point at which the vector terminates the point A . And let's suppose at the point a has coordinates a_1, a_2, a_3 .

The reason that this doesn't make much difference if you get confused is that if the vector, which originates at the origin, terminates at the point whose coordinates a_1, a_2 , and a_3 , then the components of that vector will have what? The i component will be a_1 . The j component will be a_2 . And by the way, we let k be the unit vector in the positive z direction. And the k component will be a_3 . To see this thing pictorially, notice the three-dimensional effect of the vector A . It means that if we drop a perpendicular from the head of a to the x - y plane, then you see this distance is a_1 . This distance is a_2 . And this height is a_3 .

And notice, again, in terms of adding vectors head to tail et cetera, notice that as a vector, this would be the vector $a_1 i$. This would be the vector $a_2 j$. And this would be the vector $a_3 k$. In other words, the vector a is the sum of the three vectors $a_1 i, a_2 j$, and $a_3 k$. In other words, to summarize this, the vector A is simply what? $a_1 i$, plus $a_2 j$, plus $a_3 k$. And again notice the juxtapositioning, if you wish, between the components of the vector and the coordinates of the point.

Also, we can get ahold of the magnitude of this vector. Remember, the magnitude is the length. And how can we figure out the length of this vector? Notice that, by the way, the geometry gets much tougher in three-dimensional space. But I want to show you something interesting in a minute. Let's suffer through the geometry for a minute. Let's see how we can find the length of the vector A . Well, notice, by the way, that A happens to be-- or the magnitude of A -- happens to be the hypotenuse of a right triangle. What right triangle is it? It's the right triangle that joins the origin--

let me call this point here B-- and A.

In other words, triangle OBA is a right triangle because OB is in the x-y plane and AB is perpendicular to the x-y plane. Therefore, by the Pythagorean theorem, the magnitude of a will be the square root of the square of the magnitude of OB plus the square of the magnitude of AB. On the other hand, notice that OCB is also a right triangle. So by the Pythagorean theorem, the length of OB is the square root of a_1^2 plus a_2^2 . And therefore, you see, by the Pythagorean theorem, the square of the magnitude of A is this squared plus this squared.

In other words, the magnitude of A is the square root of a_1^2 squared, plus a_2^2 squared, plus a_3^2 squared. In other words, in Cartesian coordinates, to find the magnitude of a vector, you need only take the positive square root of the sum of the squares of the components. It's a regular distance formula, the Pythagorean theorem.

By the way, just a very, very quick review from part one of our course-- remember that technically speaking, the square root is a double-valued function. Namely, any positive number has two square roots. All right? In other words, the square root of 16 is 4. It's also negative 4. But by convention, when we don't indicate a sign, we're always referring to the positive square root. And that's fine. Because after all, the magnitude-- the length of an arrow-- is always looked upon as being what? Some non-negative number. Notice that the negativeness is taken care of by the sense. OK? And this is what's rather interesting. I think it's worth noting.

If we use Cartesian coordinates in one dimension, you have is the x-axis, in which case any vector is some scale or multiple of the unit vector i . In other words, A is equal to $a_1 i$. In this case, the magnitude of A is just what? It's the square root of a_1^2 squared. And notice that means the positive square root-- notice, by the way, that that's the usual meaning of absolute value.

After all, if a_1 were already positive, taking the positive square root of its square root of its square. These tongue twisters. The positive square root of the square would

give you the number back again. On the other hand, if a_1 were negative, and then you square it, and then take the positive square root, all you do is change the sign of a_1 . And since it was negative, changing its sign makes it positive. So this is the usual absolute value, all right?

Now in terms of the previous unit, we saw that in two-dimensional space, using i and j as our basic vectors, that if A was the vector $a_1 i$ plus $a_2 j$, then the magnitude of A , again, by the Pythagorean theorem, was just a square root of a_1 squared plus a_2 squared. And now we've seen that structurally-- even though geometrically it was harder to show-- that if A in three-dimensional space is $a_1 i$ plus $a_2 j$ plus $a_3 k$, then A is what? The positive square root of a_1 squared, plus a_2 squared, plus a_3 squared.

And notice you see this structural resemblance. This is a beautiful structural resemblance. Why is it a beautiful structural resemblance? Because now maybe we're getting the feeling that once our rules and recipes are made up, we don't have to worry about how difficult the geometry is. In other words, except for the fact that you have an extra component to worry about here, there is no basic difference structurally between finding the length of a vector in one-dimensional space or in three-dimensional space.

And in fact, another reason that I'm harping on this is the fact that in a little while-- meaning within the next few lessons-- we are going to be talking about some horrible thing called n -dimensional space, which geometrically cannot be drawn, but which analytically preserves the same structure that we're talking about here. You see, the whole bag is still this idea of structure. We're trying to emphasize that the recipes remain the same independently of the dimension.

And just to carry this idea a few steps further, what we're saying is, remember in the last unit, we talked about the fact that the nice thing about i and j components was that the definition for adding two vectors was that if they were in terms of i and j components, you just add it component by component? Well again, without belaboring the point and leaving the details to the text and to the exercises, let's

keep in mind that it's relatively easy to show that the same formal definition of addition-- namely, placing the arrows head to tail, et cetera-- forgetting about Cartesian coordinates, but notice the definition of addition doesn't mention the coordinate system. It just says put the vectors head to tail, et cetera.

And what I'm saying is it's easy to show, under those conditions, that if A is the vector $a_1 i + a_2 j + a_3 k$, and B is the vector $b_1 i + b_2 j + b_3 k$, then you still add vectors the same way in three space as you did in two space. Again, the geometric arguments are a little bit more sophisticated, if only because it's harder to draw three dimensions to scale than it is two dimensions. But the idea is what? $A + B$ is just obtained by adding what? The two i components, the two j components, the two k components.

And similarly, for scalar multiplication, in the same way in terms of i and j components, you simply multiplied each component by the scale of c . If c is any number and A is still the vector $a_1 i + a_2 j + a_3 k$, the scalar multiple c times A turns out to be $c a_1 i + c a_2 j + c a_3 k$. That just multiply component by component. And by the way, don't lose track of this important point, that the scalar multiple cA means the same thing in two dimensions as it did three. Because after all, when you draw that vector, once that arrow is drawn, it's a one-dimensional thing if you pick as your axis the line of action of the particular arrow.

In other words, scalar multiplication means the same thing. What's happening is that you're looking at it in terms of different components. You're looking at it in terms of i , j , and k , rather than just i and j , or in terms of any other coordinate system. But carrying this out in particular, notice then, the connection between the vector minus B -- negative B -- and the vector -1 times B . Namely, the vector negative B , which has the same magnitude and direction as B , but the opposite sense, is the same way as saying what? Multiply B by minus 1. Because that gives you a vector which has the same magnitude-- namely 1 times as much as B -- the same direction, and the opposite sense.

In other words, the vector negative B , in three-dimensional space, is minus $b_1 i$,

minus $b_2 j$, minus $b_3 k$. And therefore, since A minus B still means A plus negative B , the vector A minus B is simply obtained how? You subtract the same way as we did in ordinary one-dimensional analytic geometry, in terms of a number line. In other words, you subtract component by component-- subtracting what? The first vector minus the corresponding component of the second vector.

In other words, in Cartesian coordinates, the vector A minus B -- once we know a_1 , a_2 , and a_3 as the components of A , b_1 , b_2 , and b_3 as the components of B , we just subtract component by component to get this particular result. And this is, I think, very important to understand. By the same token, as simple as this is, it's going to cause-- if we're not careful-- great misinterpretation. Namely, the vector arithmetic that we're talking about does not depend on Cartesian coordinates. Vectors or arrows, as we're looking at them, and the arrows exist no matter what coordinate system we're using.

What is particularly important, however, is that many of the luxuries of using vector arithmetic hinge on using Cartesian coordinates. Now what do I mean by that? Well, let me just emphasize: Note the need for Cartesian coordinates. Let me talk instead, you see, about something called polar coordinates. Now later on, we're going to talk about polar coordinates in more detail-- in fact, in very, very much more detail. But for the time being, let's view polar coordinates simply as a radar-type thing, as a range and bearing-type navigation. Namely, in polar coordinates, the way you specify a vector in the plane, say, is you specify its length, which we'll call r , and the angle that it makes with the positive x -axis that we'll call θ .

In other words, in polar coordinates, the vector A , which I've labeled r_2 comma θ_2 , that would indicate that the magnitude of A is r_2 . And that the angle that A makes with the positive x -axis is θ_2 . And when I label the vector B as r_1 comma θ_1 , it's simply my way of saying that the magnitude of the vector B is some length, which we'll call r_1 . And the angle that B makes with the positive x -axis is some angle which we'll call θ_1 . OK? Perfectly well-defined system, isn't it? In other words, if I tell you the range and bearing, I've certainly given you as much information as if I told you the i and j components.

Now here's the important point, the real kicker to this thing. And that is that the vector A minus B -- and A minus B is what? It's this vector here. And by the way, notice how vector arithmetic goes. How could you tell quickly whether this is A minus B or B minus A ? We talked about this in the previous unit, and had exercises and discussions on it. But notice again, the structure of vector arithmetic. If you look at addition, what is this vector here? This is the vector which must be added on to B to give the vector A . And that, by definition, is called what? A minus B . A minus B is what vector? The vector you must add on to B to yield A . Subtraction is still the inverse of addition.

And hopefully by now, you should be becoming much less fearful of this vector notation, that structurally, it's the same as our first unit of arithmetic. Well, numerical arithmetic, OK, the more regular type, ordinary type of arithmetic. But here's the important point. This vector is still A minus B . But I claim that it's trivial to see that this vector is not the vector whose magnitude is r_2 minus r_1 and whose angle of bearing is θ_2 minus θ_1 .

Now the word "trivial" is a very misleading word in mathematics. One of our famous mathematical anecdotes is the professor who said that a proof was trivial. The student says, "It doesn't seem trivial to me." Their professor says, "Well, it is trivial." He didn't quite see it in a hurry. He says, "Wait here." He ran down to his office. Came up three hours later with a ream of papers that he had written on, bathed in perspiration, and a big smile. And he said, "I was right. It was trivial."

So just because what I think is trivial may not be what you think is trivial, let me go through this statement in more computational detail. What I'm saying is, let v denote the vector that we called A minus B before. Notice that in terms of what the vectors A and B are, this length is r_1 . This length is r_2 . And notice that since this whole angle was θ_2 , and the angle from here to here with θ_1 , this angle in here must be the difference between those two angles, which is θ_2 minus θ_1 .

Now notice that this is still a triangle. The length of this triangle is still expressible in terms of these sides and the included angle. But you may recall from plane

trigonometry and our refresher of that in the first part of our course-- part one-- that to find the third side of a triangle given two sides and the included angle, one must use the Law of Cosines.

In other words, the magnitude of v has what property? That its square is equal to the sum of the squares of these two sides minus twice the product of these two lengths times the cosine of the included angle. In other words, this is just-- I hope this shows up all right-- this is the Law of Cosines. And if it's difficult to read, don't bother reading it. It's still the Law of Cosines, and hopefully you recognize it as such.

In other words, to find the magnitude of v , it's not just r_1 or r_2 minus r_1 . It's the square root of $(r_1^2 + r_2^2 - 2 * r_1 * r_2 * \cos(\theta_2 - \theta_1))$. Can you use polar coordinates if you want? The answer is you bet you can. But when you do use it, make sure that whenever you're going to find the magnitude, that you don't just subtract the two magnitudes. You have to use this.

And by the way, notice I haven't even gone into this part because it's irrelevant from the point of view that I'm trying to emphasize now. This mess just gives you the magnitude of a vector. It doesn't even tell you what direction it's in. I have to still do a heck of a lot of geometry if I want to find out what this angle here is-- a lot of work involved. I can use polar coordinates, but I lose some of the luxury of Cartesian coordinates. In fact, notice that the vector whose magnitude is r_2 minus r_1 and whose angle of bearing is θ_2 minus θ_1 , can also be computed.

Namely, how do we find θ_2 minus θ_1 ? Well up here, we have the angle θ_2 . We have the angle θ_1 . We computed θ_2 minus θ_1 . I can now mark off that angle here, θ_2 minus θ_1 . I can now take a circle of radius r_1 - well, why even say a circle-- mark off the length r_1 onto r_2 , assuming r_2 is the greater of the two lengths. That difference will be r_2 minus r_1 . In other words, if I swing an arc over here, this would be r_1 also. This distance here would be r_2 minus r_1 . I take that distance, and mark it off down here.

And what vector is this? This is the vector, which in polar coordinates would have its

magnitude equal to r_2 minus r_1 , and have its angle equal to θ_2 minus θ_1 . And all I want you to see is no matter how you slice it, just from this picture alone, this vector is not the same as this vector. And by the way, don't make the mistake of saying, gee, the way you've draw them, they look like they could be the same length.

Remember that even if by coincidence these two vectors had the same length, you must remember that vector equality requires not just the same magnitude, but the same direction. And even with the same direction, the same sense. What should be obvious is, at least based on this one picture, that the direction of this vector certainly is not the same as the direction of this vector.

In other words, I guess I've said this many times in part one, and even though it comes out like phony facetiousness, I mean it quite sincerely. Given two vectors in polar coordinates, one certainly has the right to invent the vector $(r_2 - r_1, \theta_2 - \theta_1)$. You certainly have the right to do that, but you don't have the right to call that A minus B . What I meant by being facetious is, you have the right to call it that, but you're going to be wrong. Because A minus B has already been defined. And all we have now is a choice of what the answer looks like in a particular coordinate system.

And by the way, just as a quick review, assuming that you've studied different number bases at one time in your careers, this has come up many times in the mathematics curriculum under the heading of "Number-versus-Numeral," or more generally, I call it the heading of "Name-versus-Concept."

For example, consider the number 6. 6 is this many, and no matter how you slice it again, this many is an even number. It breaks up into bundles of 2 with none left over. OK? If you were to write 6 as a base 5 numeral, it's what? One bundle of 5 with 1 left over. In other words, notice that the number 6 ends in an odd digit in base 5. In other words, the test for evenness by looking at the last digit depends on the number base.

In other words, in an even base, a number is even if its units digit is even. In an odd

base, a number may or may not be even, depending on whether its units digit is even. In other words, the actual test that you're using-- the actual convenient computational property-- depends on the base. But the important point is that 6 is always even, independently of a number base. In fact, the number 6 as six tally marks was invented long before place value enumeration. You see what I'm driving at? In other words, the number concept stays the same. The concept stays the same.

What the thing looks like in terms of a particular set of names, that's what depends on the names that you choose. All right? I think that is very, very important to see. The vector properties do not change. The convenient computational recipes do depend on what coordinate system you're choosing. And that's one of the reasons-- the same as in part one of this course-- why we find it very convenient whenever possible to use Cartesian coordinates. OK?

And let me just summarize this some more. Structurally, this is very important here. Structurally, arrow arithmetic is the same for both two and three dimensions. For example, without belaboring the point, whether we're dealing with two-dimensional arrows or three-dimensional arrows, $A + B$ is $B + A$. $A + B + C$ is equal to $A + B + C$, and let me just say, et cetera, and not belabor this particular point that structurally, we cannot tell the difference between two-dimensional vectors and three-dimensional vectors.

What we can tell the difference between, I guess, is the geometry. The geometry may be more difficult to visualize in three space than in two space. But structurally, we cannot tell the difference. And I guess this is what this thing is all about, that as you read the textbook and do your assignment, there is going to be a tendency on your part to try to rely on diagrams. And boy, you have to be an expert in descriptive geometry to be able to take a view of something in three dimensions, and try to get its true lengths as you look along various axes and lines of sight.

The beauty is going to be that as we continue on with this course, we will use geometry for motivation. But once we have motivated things by geometry, we will

extract those rules that we can use even in cases where we can't see the picture. In much the same way, going back to our first lecture when we talked about why b to the 0 equals 1, we made up our rules to make sure that they would conform to the nice computational recipes that were prevalent in the more simple cases.

And we are going to play this thing to the hilt. We're going to really exploit this and explore it in future lectures, and in fact, throughout the rest of this course. But more about that next time. And until next time, good bye.

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