Unit 2: An Introduction to Partial Derivatives

1. Lecture 3.020

a. Calculus of Several Variables

Given \( w = f(x, y, z) \), we let \( z = 0 \) (to take advantage of geometry) so \( w = f(x, y) \)
where \( x \) and \( y \) are independent.

Find \( y \) at \( y = 0 \)
and let \( x \) vary between \( -1, 1 \) and \( 2, 1, 3 \).

Example

\[
\frac{\partial f}{\partial x} (1, 2, 3) = z = \frac{\partial f}{\partial x} (1, 2, 3) = \frac{\partial f}{\partial x} (1, 2, 3)
\]

b. \( f_{1}, f_{2}, \ldots f_{n} \) called partial derivatives of \( f \) with \( x_{1}, x_{2}, \ldots, x_{n} \).

"usual" derivative properties still hold.

Example:

Let \( w = e^{x_{1}} \sin(2x_{2}) \);

Then \( \frac{\partial w}{\partial x_{1}} = e^{x_{1}} \cos(2x_{2}) \) \( + 2e^{x_{1}} \sin(2x_{2}) \)

Example (Explanation?)

\( w = e^{x_{1}} \cos(2x_{2}) \)

Let \( x_{1} = 2, x_{2} = 3 \).

\( w = f(2, 3) = e^{2} \cos(6) \)

\( \frac{\partial w}{\partial x_{1}} = e^{x_{1}} \cos(2x_{2}) \)

Generally we consider \( w = g(x_{1}, x_{2}) \) or \( w = h(x_{1}, x_{2}) \) by hand, even \( w = u(x, y) \)

[Think of polar vs. Cartesian Coordinates. Never use \( x(x, y) \)]

c. Pictorially (n = 2)

\[
\frac{\partial z}{\partial x} = \frac{\partial z}{\partial y}
\]

Tangent Plane

\[
\frac{\partial z}{\partial x} = \frac{\partial z}{\partial y} = \frac{\partial z}{\partial y} = \frac{\partial z}{\partial y}
\]

3.2.1
2. Read Thomas, Sections 15.2 and 15.3.

3. Exercises:

3.2.1(L)

a. If \( f(x,y) = x^2 + y^3 \), compute \( f_x(1,2) \).

b. If \( f(x,y) = x^3 y + x^4 + y^5 \), compute \( f_{xx}(1,2) \) where \( f_{xx}(1,2) \) means 
\[
\frac{\partial^2 f}{\partial x^2}
\]

\( (1,2) \).

c. Find \( f_y(1,2,3,4) \) if \( f(w,x,y,z) = w^2 xy + z^3 y^2 + x^3 zw \).

3.2.2

a. Determine \( f_w(w,x,y,z) \) and \( f_{ww}(w,x,y,z) \) if \( f(w,x,y,z) = w^3 x^2 y + x^3 y^2 z + wz^4 \). In particular, determine \( f_{ww}(1,2,3,4) \).

b. Compute \( \frac{\partial z}{\partial x} \) if 
\[
z^3 xy + z^5 y + \cos z = 1.
\]

3.2.3(L)

Let \( x \) and \( y \) be a pair of independent variables and define \( u \) and \( v \) by 
\( u = 2x - 3y \) and \( v = 3x - 4y \).

a. Show that \( u \) and \( v \) are then also a pair of independent variables.

b. Solve the above equations and express \( x \) and \( y \) in terms of \( u \) and \( v \). From this compute \( \frac{\partial x}{\partial u} \) and compare this with \( \frac{\partial u}{\partial x} \).

c. Express \( x \) in terms of \( u \) and \( y \), and then compute \( \frac{\partial x}{\partial u} \). How does this answer compare with the result in (b)?

d. Express \( u \) in terms of \( x \) and \( v \), and then compute \( \frac{\partial u}{\partial x} \). In this case, does \( \frac{\partial u}{\partial x} = \frac{1}{(\frac{\partial x}{\partial u})} \) where \( \frac{\partial x}{\partial u} \) is as in (b)?

3.2.4

Given that \( x \) and \( y \) are independent variables, define \( u \) and \( v \) by 
\( u = x^2 - y^2 \) and \( v = 2xy \).

a. Explain why \( u \) and \( y \) are also independent variables.

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3.2.4 continued

b. Show that \( \frac{\partial x}{\partial u} = \frac{1}{\frac{\partial u}{\partial x}} \).

c. Determine the value of \( \frac{\partial x}{\partial u} \).

3.2.5 (L)

From the polar coordinate relations, \( y = r \sin \theta \) and \( x = r \cos \theta \), compute \( \frac{\partial \theta}{\partial y} \) where, from now on, \( \frac{\partial \theta}{\partial y} \) will be interpreted to mean \( \frac{\partial \theta}{\partial y} \) unless otherwise specified.

3.2.6

Given that \( x \) and \( y \) are independent variables, assume that \( u \) and \( v \) are functions of \( x \) and \( y \) [we usually denote this as either \( u = u(x,y) \) and \( v = v(x,y) \), or, if we feel that misinterpretation might arise, as \( u = f(x,y) \) and \( v = g(x,y) \)] such that \( u \) and \( v \) are also independent variables. Assume further that we also know that \( u^2 = y^2 v \). Determine \( \frac{\partial u}{\partial y} \).

3.2.7 (L)

Let \( S \) be the surface defined by the Cartesian equation \( z = x^2 + y^3 \). Assume that there is a plane which is tangent to \( S \) at the point \( P(1,2,9) \). Find the equation of this plane.

3.2.8

Assuming that the surface defined by the Cartesian equation \( z = x^3y^2 + x^5 + y^7 \) has a tangent plane at the point \( (1,1,3) \), find the equation of this plane.

3.2.9 (L)

Let the surface \( S \) have the Cartesian equation \( x = g(y,z) \).

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3.2.9(L) continued

a. Assuming that S possesses a tangent plane at the point \((x_0, y_0, z_0)\), find the equation of this plane.

b. The plane M is tangent to the surface \(x = e^{3y-z}\) at the point \((1, 2, 6)\). Find the equation of M.

c. Check the solution in (b) by expressing \(x = e^{3y-z}\) in the form \(z = f(x, y)\).
Resource: Calculus Revisited: Multivariable Calculus
Prof. Herbert Gross

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