Solutions

BLOCK 3:
PARTIAL DERIVATIVES
Solutions
Block 3: Partial Derivatives

Pretest

1. Use polar coordinates to obtain \( w = \frac{2r \sin \theta (r \cos \theta)}{r^2} = \frac{2 \sin \theta \cos \theta}{r} \) if \( r \neq 0 \). Therefore, \( w = \sin 2\theta \) \( (r \neq 0) \) and therefore, \( \lim_{(x,y) \to (0,0)} w = \lim_{(r,\theta) \to (0,\theta_0)} w = \sin 2\theta_0 \) and this depends on \( \theta_0 \).

2. \( 5x + 7y + 6z = 18 \).

3. \( h''(r) + \frac{1}{r} h'(r) \).

4. \( x^2 e^y + e^x + e^y = 2 \).

5. \( \frac{1}{y + 1} \).
While it was more natural, in terms of an extension of lower dimensional cases, to invent the Euclidean metric, the point is that a metric is defined solely by the properties mentioned in this exercise. Since, from a logical point of view, we use nothing but inescapable consequences of these properties, it follows that any metric that has these same properties may be used in our study. In particular, we want to show in this exercise that the Minkowski metric has these properties. Thus,

a. Since \( \|x\| = \max\{|x_1|, \ldots, |x_n|\} \) the fact that each \( |x_i|, (i=1, \ldots, n) \), is at least as great as zero guarantees that the maximum of the \( x_i \)'s is also at least as great as zero. This proves that

\[ \|x\| \geq 0 \]

Now, if it happens that \( \|x\| \) actually equals 0, then by definition, the maximum \( |x_i| \) is also zero. But no \( |x_i| \) can be less than zero; thus, the maximum \( |x_i| \) being zero guarantees that all \( |x_i| \) equal zero or that all \( x_i = 0 \), and this means that

\[ x = (x_1, \ldots, x_n) = (0, \ldots, 0) = 0. \]

As for checking that \( \|x+y\| \leq \|x\| + \|y\| \), we see from the definition of the Minkowski metric that this is equivalent to proving that

\[ \max\{|x_1+y_1|, \ldots, |x_n+y_n|\} \leq \max\{|x_1|, \ldots, |x_n|\} + \max\{|y_1|, \ldots, |y_n|\} \]

(1)

Unless we are bothered by the fairly abstract notation, (1) is almost trivial to prove. Namely, let \( |x_k+y_k| \) denote the maximum of \( |x_1+y_1|, \ldots, \) and \( |x_n+y_n| \). We then have

\[ \max\{|x_1+y_1|, \ldots, |x_n+y_n|\} = |x_k+y_k| \]
and, since $x_k$ and $y_k$ are real numbers, we already know from part 1 of our course that $|x_k + y_k| \leq |x_k| + |y_k|$. Hence,

$$\max\{|x_1 + y_1|, \ldots, |x_n + y_n|\} \leq |x_k| + |y_k| \quad (2)$$

Finally, since $|x_k| \leq \max\{|x_1|, \ldots, |x_n|\}$ and $|y_k| \leq \max\{|y_1|, \ldots, |y_n|\}$, we have from (2) that

$$\max\{|x_1 + y_1|, \ldots, |x_n + y_n|\} \leq \max\{|x_1|, \ldots, |x_n|\} + \max\{|y_1|, \ldots, |y_n|\}.$$

In fact, if the $x$'s and $y$'s all have the same sign, equality can hold if and only if

$$|x_k| = \max\{|x_1|, \ldots, |x_n|\} \quad \text{and} \quad |y_k| = \max\{|y_1|, \ldots, |y_n|\},$$

which means that the maximum components of $x$ and $y$ "match" (i.e., have the same subscript).

For example, if $x = (2, 4, 1)$ and $y = (3, 2, 5)$, the maximum components do not match (in $x$ it is the second component, 4, and in $y$ it is the third component, 5). In this case $x+y = (5, 6, 6)$, whence $\|x+y\| = 6$, while $\|x\| + \|y\| = 4+5 = 9$. Thus $\|x+y\| < \|x\| + \|y\|$.

On the other hand if $x = (2, 5, 1)$ and $y = (3, 7, 2)$ [where the maximum components match], $\|x+y\| = \max\{5, 12, 3\} = 12$ while $\|x\| + \|y\| = 5+7 = 12$, so that $\|x+y\| = \|x\| + \|y\|$.

Finally, we must show that

$$\|a \cdot x\| = |a| \cdot \|x\|.$$

At any rate,
3.1.1(L) continued

\[ \|a \cdot x\| = \max\{|ax_1|, \ldots, |ax_n|\} \]
\[ = \max\{|a| |x_1|, \ldots, |a| |x_n|\} \]
\[ = |a| \max\{|x_1|, \ldots, |x_n|\}^* \]
\[ = |a| \|x\| \]

b. From a mechanical point of view, this is hardly a learning exercise. In fact, it is more like a trivial drill exercise. We are given that \( x = (2, 4, 1) \) while \( y = (4, 4, 5) \). (We picked \( n = 3 \) so that you could use your geometric intuition if you so desired; that is, with \( n = 3 \) we may think of \( x \) as being \( 2\hat{i} + 4\hat{j} + \hat{k} \), etc). Then,
\[ x \cdot y = (2)(4) + (4)(4) + (1)(5) = 29 \]  
(1)

On the other hand, our definition of the Minkowski metric yields that
\[ \|x\| = \max\{2, 4, 1\} = 4 \]  
(2)
and
\[ \|y\| = \max\{4, 4, 5\} = 5 \]  
(3)

From (2) and (3) we have that \( \|x\| \|y\| = 20 \), and comparing this result with (1), we have that
\[ |x \cdot y| > \|x\| \|y\| \]  
(4)

*Notice that if \( c > 0 \) is any constant, then
\[ \max\{cx_1, \ldots, cx_n\} = c \max\{x_1, \ldots, x_n\} \]
since \( c \) "magnifies" each \( x_i \) in the same ratio.
3.1.1 (L) continued

Equation (4) violates a key property of a dot product, especially as we know it in the 2- and 3-dimensional cases. In other words, the Minkowski metric does not obey the structural property that is so often used in dot products (namely, $|x \cdot y| \leq |x| |y|$) while the Euclidean metric does have this structural property (as we have proven in our supplementary notes).

This explains why most texts stress the Euclidean metric over the Minkowski metric, even though as we shall see in the next few exercises, the Minkowski metric is most helpful in our discussion of limits in the case of real functions of n-tuples, especially with $n > 3$.

In our later work (especially Block 7) we will do a great deal of work with the generalized notion of a dot product and in this context the Euclidean metric has desired properties not shared by the Minkowski metric.

From the point of view of our "game," however, the Minkowski metric is on a par with the Euclidean metric in the type of investigation we are currently undertaking where the dot product of two n-tuples is not an issue.

3.1.2

We have \( \lim_{x \to a} f(x) = L_1 \) and \( \lim_{x \to a} g(x) = L_2 \).

Given \( \epsilon > 0 \), we must find \( \delta > 0 \) such that

\[
0 < |x - a| < \delta \Rightarrow |[f(x) + g(x)] - [L_1 + L_2]| < \epsilon.
\]

Just as in the scalar case, we observe that

\[
|f(x) + g(x)| - [L_1 + L_2] = |[f(x) - L_1] + [g(x) - L_2]| \\
\leq |f(x) - L_1| + |g(x) - L_2| \tag{1}
\]

By definition of \( L_1 \) and \( L_2 \), we can find \( \delta_1 \) and \( \delta_2 \) such that
Thus if \( \delta \leq \min\{\delta_1, \delta_2\} \) it follows from (2) that

\[
0 < |x-a| < \delta \Rightarrow |f(x)-L_1| + |g(x)-L_2| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon
\]

Combining (3) with (1) yields the desired result, namely

\[
0 < ||x-a|| < \delta \Rightarrow |(f(x)+g(x)) - [L_1+L_2]| < \epsilon
\]

Note:

In this exercise, we did not have to specify whether it was the Euclidean metric or the Minkowski metric that was being used. The point is that our proof used only those properties that were true for either metric. To be sure, the actual value of \( \delta \) for a given value of \( \epsilon \) might depend on which metric was used, but the existence of \( \delta \) did not.

On the other hand, there may be a tendency to think in terms of the Minkowski metric when we say "let \( \delta = \min\{\delta_1, \delta_2\} \)." The point is that while this part of our logical strategy is the same whether it is the Euclidean metric or the Minkowski metric which is being used, what is true is that the motivation behind the strategy is the same as that which inspired the "invention" of the Minkowski metric. Namely, in both instances, the idea is that to make sure that something is sufficiently small, we make sure that its maximum dimension is sufficiently small, regardless of how we define "dimension."

One final word that may be of interest is that in our 1-dimensional study of calculus, that is, when we were studying limits of functions of a single real variable, there was no distinction between the Euclidean and the Minkowski metric. The interesting point is that
in the special case of n=1, the two metrics are identical. Namely, for a real number, $x$, $|x|$ and $\sqrt{x^2}$ are the same (and, obviously, for the case of only one component, $|x| = \max(|x|)$). That is, either the Minkowski or the Euclidean metric would have led to the same definition of magnitude in the case of a 1-dimensional space.

3.1.3(L)

a. We must find $\delta$ such that

$$0 < \|(x,y)-(2,3)\| < \delta + |x^2+y^3-31| < \varepsilon$$

As seen in our supplementary notes, $|x^2+y^3-31| < \varepsilon$ if

$$|x^2-4| < \frac{\varepsilon}{2} \quad \text{and} \quad |y^3-27| < \frac{\varepsilon}{2}$$

To make $|x^2-4| < \frac{\varepsilon}{2}$ we take into consideration that $x$ is near 2 (i.e., $\varepsilon$ is small). That is, we may assume that

$$2-\xi < x < 2 + \xi \quad \text{where} \quad |\xi| < 1$$

In particular, then,

$$1 < x < 3$$

and therefore

$$3 < x+2 < 5 \quad \text{(1)}$$

Thus, near $x=2$,

$$|x^2-4| = |x-2||x+2| = |x-2|(x+2)$$

or, from (1)

$$|x^2-4| < 5|x-2| \quad \text{(2)}$$

From (2) it is easily seen that $|x-2| < \frac{\varepsilon}{10}$ implies that

$$|x^2-4| < \frac{\varepsilon}{2}.$$
Thus, the $\delta_1$ referred to in our notes may be taken to be $\frac{\varepsilon}{10}$. In other words,

$$0 < |x-2| < \frac{\varepsilon}{10} \Rightarrow |x^2-4| < \frac{\varepsilon}{2}$$

(3)

As for making $|y^3-27| < \frac{\varepsilon}{2}$, we observe that

$$y^3-27 = (y-3)(y^2+3y+9).$$

Hence

$$|y^3-27| = |y-3||y^2+3y+9|$$

(4)

Since we know that $y$ is near 3, it follows that $y^2+3y+9$ is near 27 (because we already know that $y^2+3y+9$ is a continuous function of $y$). In particular, we shall assume $y$ is close enough to 3 so that $y^2+3y+9 (=|y^2+3y+9|) < 28$.

With this in mind, (4) becomes

$$|y^3-27| = |y^2+3y+9||y-3|< 28|y-3|$$

(5)

From (5), it follows immediately that if $|y-3| < \frac{\varepsilon}{56}$ then $|y^3-27| < \frac{\varepsilon}{2}$. In other words

$$0 < |y-3| < \frac{\varepsilon}{56} \Rightarrow |y^3-27| < \frac{\varepsilon}{2},$$

and $\frac{\varepsilon}{56}$ is the $\delta_2$ of our notes.

Since $\varepsilon$ is positive, it is clear that $\frac{\varepsilon}{56} < \frac{\varepsilon}{10}$. Hence, if

$$\delta = \min\{\delta_1, \delta_2\}$$

we have

$$\delta = \frac{\varepsilon}{56}.$$
In summary, using the Minkowski metric, given $\varepsilon > 0$ choose $\delta = \frac{\varepsilon}{56}$. Then

$$0 < \|(x,y) - (2,3)\| < \delta \Rightarrow |x^2 + y^3 - 31| < \varepsilon$$

[and by our remarks in the supplementary notes, the value of $\delta$ also suffices if we are using the Euclidean metric, as we shall review in part (b)].

b. Pictorially, we have

Since $|x^2 + y^3 - 31|$ is true for all $(x,y)$ in rectangle ABCD, it is, in particular, true for all $(x,y)$ in the circle centered at $(2,3)$ with radius $\frac{\varepsilon}{56}$. The equation of this circle is

$$(x-2)^2 + (y-3)^2 = \left(\frac{\varepsilon}{56}\right)^2.$$ 

In any event

$$\sqrt{(x-2)^2 + (y-3)^2} < \frac{\varepsilon}{56} \Rightarrow |x^2 + y^3 - 31| < \varepsilon$$
Hence,

\[ 0 < \|(x,y)-(2,3)\| < \frac{\varepsilon}{56} + |x^2+y^3-31| < \varepsilon \]

is true even if \(\|(x,y)-(2,3)\|\) now denotes the Euclidean metric.

We have \(\mathbf{x} = (x_1,x_2,x_3,x_4), \mathbf{l} = (1,1,1,1)\) and \(f(\mathbf{x})=x_1^2+2x_2+x_3^3+x_4^2\).

We must prove that

\[
\lim_{\mathbf{x} \to \mathbf{l}} f(\mathbf{x}) = f(\mathbf{l})
\] (1)

Since \(\mathbf{l} = (1,1,1,1)\), the definition of \(f\) tells us that

\[ f(\mathbf{l}) = f(1,1,1,1) = 1^2+2(1) + 1^3 + 1^2 = 5. \]

Thus, we see that \(f(\mathbf{x})\) exists at \(\mathbf{l}\) and its value is 5. Next we must show that

\[
\lim_{\mathbf{x} \to \mathbf{l}} (x_1^2+2x_2+x_3^3+x_4^2) = 5
\] (2)

Equation (2), by definition, now implies that for a given \(\varepsilon>0\) we must find \(\delta>0\) such that

\[ 0 < \|\mathbf{x}-\mathbf{l}\| < \delta \Rightarrow |x_1^2+2x_2+x_3^3+x_4^2-5| < \varepsilon \] (3)

Up to now we have not specified what metric we are using. If we specify the Minkowski metric (noting that the same \(\delta\) will also work for the Euclidean metric), (3) becomes

Find \(\delta>0\) such that

\[ \max\{|x_1-1|, |x_2-1|, |x_3-1|, |x_4-1|\} < \delta \Rightarrow |x_1^2+2x_2+x_3^3+x_4^2-5| < \varepsilon \] (4)
We rewrite \( |x_1^2 + 2x_2 + x_3^3 + x_4^2 - 5| \) in the more suggestive form

\[
| (x_1^2 - 1) + 2(x_2 - 1) + (x_3^3 - 1) + (x_4^2 - 1) |
\]

(since this is how we got 5 in the first place), where upon we may conclude that

\[
| x_1^2 + 2x_2 + x_3^3 + x_4^2 - 5 | \leq | x_1^2 - 1 | + 2 | x_2 - 1 | + | x_3^3 - 1 | + | x_4^2 - 1 | \quad (5)
\]

From (5), \( | x_1^2 + 2x_2 + x_3^3 + x_4^2 - 5 | \) will be less than \( \varepsilon \) as soon as

\[
| x_1^2 - 1 | + 2 | x_2 - 1 | + | x_3^3 - 1 | + | x_4^2 - 1 | < \varepsilon.
\]

This, in turn, will happen as soon as each of our four summands is less than \( \frac{\varepsilon}{4} \). That is, we need only make each of the numbers \( | x_1^2 - 1 |, 2 | x_2 - 1 |, | x_3^3 - 1 |, \) and \( | x_4^2 - 1 | \) less than \( \frac{\varepsilon}{4} \). In other words we must have:

\[
\begin{align*}
(1) & \quad | x_1^2 - 1 | < \frac{\varepsilon}{4} \\
(2) & \quad 2 | x_2 - 1 | < \frac{\varepsilon}{4} \quad \text{(or } | x_2 - 1 | < \frac{\varepsilon}{8} \text{)} \\
(3) & \quad | x_3^3 - 1 | < \frac{\varepsilon}{4} \\
(4) & \quad | x_4^2 - 1 | < \frac{\varepsilon}{4}
\end{align*}
\quad (6)
\]

The key now is that each of the four inequalities in (6) involves only scalars. In particular, since we know that

\[
\lim_{x_1 \to 1} x_1^2 = 1
\]

\[
\lim_{x_2 \to 1} x_2 = 1
\]

\[
\lim_{x_3 \to 1} x_3^3 = 1
\]

\[
\lim_{x_4 \to 1} x_4^2 = 1
\]
it follows that we can find $\delta_1$, $\delta_2$, $\delta_3$, and $\delta_4$ such that

\[
\begin{align*}
0 < |x_1 - 1| &< \delta_1 \Rightarrow |x_1^2 - 1| < \frac{\varepsilon}{4} \\
0 < |x_2 - 1| &< \delta_2 \Rightarrow |x_2^2 - 1| < \frac{\varepsilon}{8} \\
0 < |x_3 - 1| &< \delta_3 \Rightarrow |x_3^2 - 1| < \frac{\varepsilon}{4} \\
0 < |x_4 - 1| &< \delta_4 \Rightarrow |x_4^2 - 1| < \frac{\varepsilon}{4}
\end{align*}
\]

Hence, if we let $\delta = \min\{\delta_1, \delta_2, \delta_3, \delta_4\}$, we may conclude from (7) that if $|x_i - 1| < \delta$ $(i = 1, 2, 3, 4)$ then

\[
|x_1^2 - 1| + 2|x_2 - 1| + |x_3^3 - 1| + |x_4^2 - 1| < \varepsilon
\]

and from (5) this guarantees that

\[
|x_1^2 + 2x_2 + x_3^3 + x_4^2 - 5| < \varepsilon.
\]

In other words, from the definition of the Minkowski metric,

\[
0 < |x - 1| < \delta + \max\{|x_1 - 1|, |x_2 - 1|, |x_3 - 1|, |x_4 - 1|\} < \delta \Rightarrow
\]

\[
|x_i - 1| < \delta, \ i = 1, 2, 3, 4 \Rightarrow
\]

\[
|x_1^2 + 2x_2 + x_3^3 + x_4^2 - 5| < \varepsilon
\]

which was precisely what had to be shown.

A major aim of this exercise is to have you become convinced that the idea of a metric applies to higher dimensional cases (in our example, we used it in the 4-dimensional case), and that the Minkowski metric allows us to find an $n$-dimensional distance in terms of $n$ 1-dimensional distances.
Given $\epsilon > 0$ we wish to determine $\delta$, in terms of $\epsilon$ such that

$$|x-1| < \delta \Rightarrow |x_1^2 + 2x_2 + x_3^3 + x_4^2 - 5| < \epsilon.$$  

We may take equation (6) of the previous exercise as our starting point.

Namely, near $x_1 = 1$, $x_1 + 1$ is near 2. In particular, we may assume $|x_1 + 1| = x_1 + 1 < 3$. Then

$$|x_1^2 - 1| = |x_1 + 1||x_1 - 1| < 3|x_1 - 1|$$

Hence, if we make sure that $|x_1 - 1| < \frac{\epsilon}{12}$, we obtain

$$|x_1^2 - 1| < \frac{3\epsilon}{12} = \frac{\epsilon}{4}.$$  

In summary, with $\delta_1 = \frac{\epsilon}{12}$, we have

$$0 < |x_1 - 1| < \delta_1 \Rightarrow |x_1^2 - 1| < \frac{\epsilon}{4}$$

and (1) of Equation (6) in the previous exercise is obeyed.

Next, with respect to (2) of Equation (6), we want $|x_2 - 1| < \frac{\epsilon}{8}$, but now it is trivial to see that $\delta_2 = \frac{\epsilon}{8}$, allows us to say

$$0 < |x_2 - 1| < \delta_2 \Rightarrow |x_2 - 1| < \frac{\epsilon}{8}.$$  

As for (3) of Equation (6), we first write

$$x_3^3 - 1 = (x_3 - 1)(x_3^2 + x_3 + 1)$$

when $x_3$ is near 1, $x_3^2 + x_3 + 1$ is near 3. Again, we pick our interval small enough so that $|x_3^2 + x_3 + 1| = x_3^2 + x_3 + 1 < 4$. Once this is done we have

$$|x_3^3 - 1| = |x_3 - 1||x_3^2 + x_3 + 1| < 4|x_3 - 1|$$
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3.1.5 continued

and if we now make sure that \(|x_3-1| < \frac{\varepsilon}{16}\), we have \(|x_3^3-1| < 4 \left(\frac{\varepsilon}{16}\right) = \frac{\varepsilon}{4}\), and (3) is obeyed.

In summary, if we let \(\delta_3 = \frac{\varepsilon}{16}\) then

\[0 < |x_3-1| < \delta_3 \Rightarrow |x_3^3-1| < \frac{\varepsilon}{4}\]

Finally, we treat (4) of Equation (6) by observing that near 1, \(x_4+1\) is near 2, hence, for a sufficiently small neighborhood of 1, we may conclude that \(|x_4+1| = x_4+1 < 3\). Then,

\[|x_4^2-1| = |x_4+1||x_4-1| < 3|x_4-1|\]

so that if \(|x_4-1| < \frac{\varepsilon}{12}\), then \(|x_4^2-1| < \frac{\varepsilon}{4}\). That is, in this case we may let \(\delta_4 = \frac{\varepsilon}{12}\).

Letting \(\delta = \min\{\delta_1, \delta_2, \delta_3, \delta_4\}\), we have \(\delta = \left\{\frac{\varepsilon}{12}, \frac{\varepsilon}{8}, \frac{\varepsilon}{16}, \frac{\varepsilon}{12}\right\}\), and since \(\varepsilon > 0\), the fraction with the greatest denominator will be the minimum.

Hence, we may let

\[\delta = \frac{\varepsilon}{16}\]

That is

\[0 < \|(x_1, x_2, x_3, x_4)-(1, 1, 1, 1)\| < \frac{\varepsilon}{16} \Rightarrow |x_1^2+2x_2+x_3^3+x_4^2-5| < \varepsilon\]

The key point is that \(\delta\) was found by looking at four 1-dimensional limits, and it is in this sense that higher dimensional problems are as real as the lower dimensional ones. Moreover, for every \(x\) such that \(\|x-1\| < \frac{\varepsilon}{16}\), \(|f(x)-5| < \varepsilon\).

(Note: If you have no quantitative feeling for \(\delta = \frac{\varepsilon}{16}\) in this exercise, it might be worthwhile to carry the computational details a bit further. Namely, \(\delta = \frac{\varepsilon}{16}\) describes the domain
3.1.5 continued

\[
\begin{align*}
1 - \frac{\varepsilon}{16} < x_1 < 1 + \frac{\varepsilon}{16} \\
1 - \frac{\varepsilon}{16} < x_2 < 1 + \frac{\varepsilon}{16} \\
1 - \frac{\varepsilon}{16} < x_3 < 1 + \frac{\varepsilon}{16} \\
1 - \frac{\varepsilon}{16} < x_4 < 1 + \frac{\varepsilon}{16}
\end{align*}
\]

i.e., a "4-dimensional cube"

Assuming that \(\varepsilon > 0\) is sufficiently small to insure that all numbers in the above inequalities are positive, we have

\[
\begin{align*}
1 - \frac{\varepsilon}{8} + \frac{\varepsilon^2}{256} < x_1^2 < 1 + \frac{\varepsilon}{8} + \frac{\varepsilon^2}{256} \\
2 - \frac{\varepsilon}{8} < 2x_2 < 2 + \frac{\varepsilon}{8} \\
1 - \frac{3\varepsilon}{16} + \frac{9\varepsilon^2}{256} - \frac{\varepsilon^3}{4096} < x_3^3 < (1 - \frac{\varepsilon}{16})^3 < (1 + \frac{\varepsilon}{16})^3 \\
= 1 + \frac{3\varepsilon}{16} + \frac{9\varepsilon^2}{256} + \frac{\varepsilon^3}{4096} \\
1 - \frac{\varepsilon}{8} + \frac{\varepsilon^2}{256} < x_4^2 < 1 + \frac{\varepsilon}{8} + \frac{\varepsilon^2}{256}
\end{align*}
\]

Therefore

\[
\left(1 - \frac{\varepsilon}{8} + \frac{\varepsilon^2}{256} + 2 - \frac{\varepsilon}{8} + 1 - \frac{3\varepsilon}{16} + \frac{9\varepsilon^2}{256} - \frac{\varepsilon^3}{4096} + 1 - \frac{\varepsilon}{8} + \frac{\varepsilon^2}{256}\right)
\]

is a lower bound for

\[
x_1^2 + 2x_2 + x_3^3 + x_4^2
\]

in this case, while

\[
\left(1 + \frac{\varepsilon}{8} + \frac{\varepsilon^2}{256} + 2 + \frac{\varepsilon}{8} + 1 + \frac{3\varepsilon}{16} + \frac{9\varepsilon^2}{256} + \frac{\varepsilon^3}{4096} + 1 + \frac{\varepsilon}{8} + \frac{\varepsilon^2}{256}\right)
\]

is an upper bound.
Simplifying we find that $x_1^2 + 2x_2 + x_3^3 + x_4^2$ is between

$$5 - \frac{9\varepsilon}{16} + \frac{11\varepsilon^2}{256} - \frac{\varepsilon^3}{4096}$$

(1)

and

$$5 + \frac{9\varepsilon}{16} + \frac{11\varepsilon^2}{256} + \frac{\varepsilon^3}{4096}$$

(2)

Our claim now is that for $\varepsilon$ sufficiently small the numbers named by (1) and (2) be between $5-\varepsilon$ and $5+\varepsilon$. For example, assuming that $\varepsilon < 1$, we have $\varepsilon < \varepsilon^2 < \varepsilon^3$, whereupon

$$9\varepsilon + \frac{11\varepsilon^2}{256} + \frac{\varepsilon^3}{4096} < \frac{9\varepsilon}{16} + \frac{11\varepsilon}{256} + \frac{\varepsilon^3}{4096} = \frac{\varepsilon}{4096} \left(9(256) + 11(16) + 1\right)$$

Therefore,

$$\frac{9\varepsilon}{16} + \frac{11\varepsilon^2}{256} + \frac{\varepsilon^3}{4096} < \frac{\varepsilon}{4096} (2304 + 176 + 1) < \varepsilon$$

(3)

[Actually, by choosing $\delta$ to be the minimum of $\delta_1$, $\delta_2$, $\delta_3$, and $\delta_4$ we narrowed the interval and this is why $\frac{2481\varepsilon}{4096}$ occurred rather than something closer in value to $\varepsilon$.]

In any event, (3) shows that

$$5 + \frac{9\varepsilon}{16} + \frac{11\varepsilon^2}{256} + \frac{\varepsilon^3}{4096} < 5 + \varepsilon).$$

3.1.6(a)

a. Many of the same questions that could have been asked about limits in our study of calculus of a single variable can also be asked now. Among other things, the form 0/0 enters into things here just as before. Part (a) of this exercise is meant to emphasize that $f(x,y)$ cannot be continuous at $(0,0)$ since it is not even defined there.
In still other words, if \( f \) were continuous at \((0,0)\), it would mean that

\[
\lim_{(x,y) \to (0,0)} f(x,y) = f(0,0)
\]

but this is impossible since \( f(0,0) \) is not defined \((0/0)\).

b. Even though \( f \) is not defined at \((0,0)\), it does not prevent our writing \( \lim_{(x,y) \to (0,0)} f(x,y) \) since then we are interested in what happens to \( f \) as \((x,y)\) gets arbitrarily close to \((0,0)\) without equaling \((0,0)\). The problem is that the limit might (and in this case, it does) depend on just how \((x,y)\) approaches \((0,0)\). For example, suppose both \( x \) and \( y \) are not equal to 0 and we let \((x,y)\) approach \((0,0)\) along the indicated path:

![Figure 1](image)

In terms of the indicated path, we first hold \( y \) constant and let \( x \) approach 0. As soon as our path reaches the \( y \)-axis we hold \( x \) constant \((x=0)\) and let \( y \) approach 0. Computing the limit this way means

\[
\lim_{y \to 0} \left[ \lim_{x \to 0} f(x,y) \right] \tag{1}
\]
3.1.6(L) continued

Notice that (1) is a special way in which we may write

\[ \lim_{(x,y) \to (0,0)} f(x,y). \]

That is, (1) indicates but one of the infinitely many paths by which \((x,y)\) may approach \((0,0)\). Evaluating (1) yields

\[ \lim_{y \to 0} \left[ \lim_{x \to 0} \frac{x^2 - y^2}{x^2 + y^2} \right] = \lim_{y \to 0} (-1) = -1 \]  \hspace{1cm} (2)

On the other hand, the path

\[ \frac{y}{x} \]

suggests

\[ \lim_{x \to 0} \left[ \lim_{y \to 0} \frac{x^2 - y^2}{x^2 + y^2} \right] = \lim_{x \to 0} \frac{x^2}{x^2} = 1 \]  \hspace{1cm} (3)

A comparison of (2) and (3) shows that the value of

\[ \lim_{(x,y) \to (0,0)} f(x,y) \]

in this example depends on the path by which we approach \((0,0)\); since even for the two special paths we considered, we got two different answers.

The point is that when we say

\[ \lim_{x \to a} f(x) = L \]
3.1.6(L) continued

we mean the limit exists and is L for every possible path by which \( x \to a \). (In higher dimensions it means that when we make \(|x_1-a_1|,...,\) and \(|x_n-a_n|\) small, our numerical answer must be independent of this order in which we make these quantities small, and this, in turn, says that our answer is uniquely determined once \( x_k \) is sufficiently close to \( a_k \) for \( k = 1,...,n \)).

From a more positive point of view, if we are told that

\[
\lim_{(x,y) \to (a,b)} f(x,y)
\]

exists, then we can be sure that its value, among other ways, is given by either

\[
\lim_{y \to b} \left( \lim_{x \to a} f(x,y) \right) \quad \text{or} \quad \lim_{x \to a} \left( \lim_{y \to b} f(x,y) \right).
\]

The main caution is that unless we know the limit exists, these two different limits may yield different answers, as in the case of the present exercise.

c. Here we try to show what goes wrong near the origin by utilizing polar coordinates. (Among other things, this shows us that polar coordinates have a purpose quite apart from central force fields!)

Letting \( x = r \cos \theta \) and \( y = r \sin \theta \) we see that

\[
\frac{x^2 - y^2}{x^2 + y^2} = \frac{r^2 (\cos^2 \theta - \sin^2 \theta)}{r^2} = \cos^2 \theta - \sin^2 \theta, \quad \text{provided } r \neq 0
\]

\[
= \cos 2\theta, \quad \text{provided } r \neq 0
\]
3.1.6(L) continued

In other words, computing
\[
\lim_{(x,y) \to (0,0)} \frac{x^2 - y^2}{x^2 + y^2}
\]
is equivalent to computing \( \lim_{r \to 0} \cos \theta \).

The point is that the value of \( \cos 2\theta \) depends only on \( \theta \) — not \( r \).

For example, if we choose the ray \( \theta = \frac{\pi}{6} \), then \( \cos 2\theta = \frac{1}{2} \). Therefore, if \( (x,y) \to (0,0) \) along the ray \( \theta = \frac{\pi}{6} \) then
\[
\lim_{(x,y) \to (0,0)} f(x,y) = \frac{1}{2}
\]

As a check, notice that in Cartesian coordinates the ray \( \theta = \frac{\pi}{6} \) becomes the half-line \( \frac{y}{x} = \tan \frac{\pi}{6} \) in the first quadrant. Since \( \tan \frac{\pi}{6} = \frac{1}{\sqrt{3}} \), we have \( y = \frac{x}{\sqrt{3}} \) on this ray. Thus,
\[
\frac{x^2 - y^2}{x^2 + y^2} = \frac{x^2 - \frac{x^2}{3}}{x^2 + \frac{x^2}{3}} = \frac{\frac{2}{3} x^2}{\frac{4}{3} x^2} = \frac{1}{2} \text{ if } x \neq 0
\]

Therefore,
\[
\lim_{(x,y) \to (0,0)} \left[ \frac{x^2 - y^2}{x^2 + y^2} \right] = \frac{1}{2}, \text{ as predicted.}
\]

Pictorially,
3.1.6 (L) continued

If \((x,y) \neq (0,0)\), \(f(x,y)\) is constant on each ray \(\theta = \theta_0\), but the value of the constant depends on \(\theta_0\). Namely, the constant is \(\cos 2\theta_0\); i.e., for each point \((x,y)\) on the line \(\theta = \theta_0\),
\[ f(x,y) = \cos 2\theta_0. \]

\((0,0)\) is excluded from consideration on \(\theta = \theta_0\), since only if \((x,y) \neq (0,0)\].

(Figure 2)

In summary, here, we have a rather exceptional example in which we can make \(f(x,y)\) take on any value in \([-1,1]\) merely by choosing the appropriate path by which we allow \((x,y) \to (0,0)\).

3.1.7

a. We have

\[ f(x,y) = \frac{2xy}{x^2 + y^2} \] (1)

Letting \(x = r \cos \theta\) and \(y = r \sin \theta\), we have that
\[ f(x,y) = f(r \cos \theta, r \sin \theta) = \frac{2r \cos \theta (r \sin \theta)}{r^2} = 2 \sin \theta \cos \theta, \text{ provided } r \neq 0 \]

therefore
\[ f(x,y) = \sin 2 \theta, \text{ provided } r \neq 0 \]
therefore

\[
\lim_{{(x,y) \to (0,0)}} f(x,y) = \lim_{{r \to 0}} \sin 2\theta = \sin 2\theta_0 \quad (2)
\]

Since the path is determined by \( \theta_0 \), (2) shows that different paths lead to different values for the limit. This will be explored in more detail in the remaining parts of this exercise and also in the next exercise.

b. If \( \theta_0 = \frac{\pi}{4} \), (2) yields

\[
\lim_{{(x,y) \to (0,0)}} f(x,y) = \sin 2 \left( \frac{\pi}{4} \right) = \sin \frac{\pi}{2} = 1 . \quad (3)
\]

As a check \( \theta_0 = \frac{\pi}{4} \) corresponds to the ray \( y=x \), \( x>0 \). In this event, we have

\[
f(x,y) = \frac{2x^2}{x^2+x^2} = 1 \quad (\text{if } x \neq 0).
\]

Therefore, on \( y=x \), \( x>0 \), we have

\[
\lim_{{(x,y) \to (0,0)}} f(x,y) = 1
\]

which checks with (3).

c. Along the ray \( \theta=0 \), (2) yields

\[
\lim_{{(x,y) \to (0,0)}} f(x,y) = \sin 2(0) = 0
\]

As a check this is the ray \( y=0 \), \( x>0 \). In this event
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3.1.7 continued

\[
\frac{2xy}{x^2+y^2} = \frac{0}{x^2}
\]

therefore

\[
\lim_{(x,y) \to (0,0)} \left( \frac{2xy}{x^2+y^2} \right) = 0
\]

When \( \theta = \frac{\pi}{2} \), \( 2\theta = \pi \) and \( \sin 2\theta = 0 \). Therefore, on the ray \( x=0 \), \( y>0 \);

\[
\frac{2xy}{x^2+y^2} = \frac{0}{y^2}
\]

therefore,

\[
\lim_{(x,y) \to (0,0)} \frac{2xy}{x^2+y^2} = 0 \quad \text{in this case also.}
\]

Comparing (b) and (c) we see that while \( \lim_{(x,y) \to (0,0)} f(x,y) = 0 \)

if \( (x,y) \to (0,0) \) either along the x-axis or the y-axis,

\[
\lim_{(x,y) \to (0,0)} f(x,y) = 1 \quad \text{along the line of approach } \theta = \frac{\pi}{4}.
\]

This shows that (1) \( \lim_{(x,y) \to (0,0)} f(x,y) \) does not exist since its value depends on the path by which \( (x,y) \to (0,0) \), and as an aside, (2) merely knowing that \( f(x,y) \) approaches the same limit as \( (x,y) \) approaches \( (0,0) \) along either the x- or the y-axis is not enough to say that this common value is \( \lim_{(x,y) \to (0,0)} f(x,y) \).

3.1.8(L)

In a way, this exercise is a corollary to the previous one, but it is of enough significance in terms of what comes next in our course that we wanted to present the idea here.
3.1.8(L) continued

In our later work, especially with 2-tuples, we shall select two paths (four, counting sense) by which \((x,y)\) will approach \((a,b)\), and we shall try to do almost everything else in terms of what happens along these two paths. The paths will be (1) we hold \(y\) constant and let \(x\) approach \(a\), then we hold \(x\) constant and let \(y\) approach \(b\); (2) we hold \(x\) constant and let \(y\) approach \(b\), then we hold \(y\) constant and let \(x\) approach \(a\). These two paths are indicated below.

![Diagram of paths](image)

Now, as we said in Exercises 3.1.6 and 3.1.7, the limit of \(f(x,y)\) as \((x,y)\) approaches \((a,b)\) may be different along these two paths. The point is that it should not be surprising that the limit depends on the path as well as the point of evaluation of the limit; yet, as we get further into the course, we shall become more and more preoccupied with the two special paths discussed above. What will happen is that we shall have, in general, sufficiently smooth functions so that what happens along these two special paths will govern what happens along any other path, but barring this "sufficient smoothness," there is no guarantee that what happens along these two special paths is enough to tell us what happens along every path.

This exercise is meant as a forewarned; it is designed to make sure that we understand that when we say "independent of the path" we mean more than just either the horizontal or vertical directions.
Based on the fact that \( f \) as defined in the previous exercise was not defined at \((0,0)\), coupled with the fact that the limit of \( f \) as \((x,y)\) approached \((0,0)\) along the rays \( \theta = 0 \) and \( \theta = \frac{\pi}{2} \) was 0, we are motivated to define a new function \( g \) by:

\[
g(x,y) = \begin{cases} 
\frac{2xy}{x^2+y^2} & \text{if } (x,y) \neq (0,0) \\
0 & \text{if } (x,y) = (0,0)
\end{cases} \tag{1}
\]

In other words, the \( g \) of this exercise and the \( f \) of the previous exercise are identical provided that \((x,y) \neq (0,0)\), but at \((0,0)\) \( f \) is not defined while \( g \) is defined, and, in particular, \( g(0,0) = 0 \).

a. We already know from the previous exercise that \( g \) is not continuous at \((0,0)\), since among other things we showed that if \((x,y)\) approached \((0,0)\) along the ray \( \theta = \frac{\pi}{4} \), then

\[
\lim_{(x,y) \to (0,0)} \frac{2xy}{x^2+y^2} = 1 \tag{2}
\]

Notice that the meaning of "limit" in (2) tells us that \((x,y) \neq (0,0)\), and under this condition, (1) tells us that

\[
\frac{2xy}{x^2+y^2} = g(x,y).
\]

Hence, (2) may be replaced by

\[
\lim_{(x,y) \to (0,0)} g(x,y) = 1 \neq g(0,0) \text{, since } g(0,0) = 0. \tag{3}
\]
3.1.8(L) continued

Thus, at least along the path defined by $\theta = \frac{\pi}{4}$.

$$\lim_{(x,y) \to (0,0)} g(x,y) \neq g(0,0) \quad \cdots \quad (4)$$

Equation (4) tells us that $g$ is not continuous at $(0,0)$ since if it were

$$\lim_{(x,y) \to (0,0)} g(x,y) = g(0,0)$$

along every path connecting to $(0,0)$.

b. What we want to show here is that

$$\lim_{(x,y) \to (0,0)} g(x,y) = g(0,0)$$

if the approach is along either axis.

If we approach $(0,0)$ along the $x$-axis, $y=0$ for the entire path.

Thus we obtain, along this path,

$$\lim_{(x,y) \to (0,0)} f(x,y) = \lim_{(x,y) \to (0,0)} \frac{2xy}{x^2+y^2}$$

$$= \lim_{x \to 0} \frac{2xy}{x^2+y^2} \quad \text{y=0}$$

$$= \lim_{x \to 0} \frac{0}{x^2}$$

$$= 0$$

$$= g(0,0) \quad \cdots \quad (5)$$

*Recall that $x \to 0$ includes both $x \to 0^+$ and $x \to 0^-$. In terms of polar coordinates, we must check that the limits exist and are equal for the two rays $\theta=0$ and $\theta=\pi$. In this case, since $g(x,y)$ is $\sin 2\theta$ for $(x,y) \neq (0,0)$, but $\theta=0$ and $\theta=\pi$ make $g(x,y) = 0$ as (5) indicates.
Similarly, approaching \((0,0)\) along the \(y\)-axis yields

\[
\lim_{(x,y) \to (0,0)} g(x,y) = \lim_{y \to 0} \frac{2xy}{x^2 + y^2} = \lim_{y \to 0} \frac{0}{y^2} = 0 = g(0,0)
\]

This example shows once and for all that if

\[
\lim_{(x,y) \to (0,0)} g(x,y) = g(0,0)
\]

as \((x,y) \to (0,0)\) along either axis, we cannot guarantee that \(g\) is continuous at \((0,0)\). (Trivially, of course, if we know that \(g\) is continuous at \((0,0)\) then in particular (6) must be true. That is, once we know that \(\lim_{(x,y) \to (0,0)} g(x,y) = g(0,0)\) for every path, then we know that it's true for a particular path; if, however, we know what the limit is for some particular path, we cannot necessarily conclude what it is for every path.)

Our main aim here is to show how much more complicated functions of several variables are than functions of a single variable. We have deliberately chosen the case \(n=2\) here so that (1) we can again capitalize on any geometric interpretations and (2) we can show that the complications do not require that we have more than the "usual" 3-dimensional space.

The proof is not really too difficult once we get the hang of things. (If you have difficulty visualizing our approach, perhaps if, as we proceed, you refer to Figures 1 and 2, things will seem clearer.) To begin with, if \(f\) happens to be a constant function
(i.e., \( f(x_1, y_1) = f(x_2, y_2) \) for every pair of points \((x_1, y_1)\) and \((x_2, y_2)\) in the domain of \( f \)), the assertion of the theorem follows trivially, for in this case there is a number \( c \) such that \( f(x, y) = c \) for all \((x, y)\) in the domain of \( f \).

Thus, the interesting case is when \( f \) is not a constant function. Assuming, therefore, that \( f \) is not a constant function, there are at least two points \( P(x_1, y_1) \) and \( Q(x_2, y_2) \) for which \( f(P) \neq f(Q) \). Now pick any number, subject only to the condition that it lie between \( f(P) \) and \( f(Q) \). That there are such numbers is guaranteed by the fact that \( f(P) \neq f(Q) \). Let us designate one such number by \( m \). Let \( C \) denote any curve that connects \( P \) and \( Q \). Since \( f \) is continuous, there must be some point on \( C \), say \( X \), for which \( f(X) = m \), since a continuous function takes on all its intermediate values.

Of course, while you are likely to accept this last statement as being true (or at least, as being reasonable), the fact remains that we proved it only in the case of a single real variable. This proof can be extended to the case of functions of several real variables, but it is beyond our present purposes to become so computationally involved at this time. While the interested reader should feel free to prove this result on his own if he so desires, we will use a geometrical interpretation. Recall that in this case the domain of \( f \) is the entire xy-place. Now suppose that \( f \) is continuous. This means, pictorially, that the surface \( z = f(x, y) \) is unbroken. Again, to add concreteness to our discussion, let us replace such literal points as \( P \) and \( Q \) by the more specific points, say, \( P(1,2) \) and \( Q(3,4) \). Suppose, without loss of generality, that \( f(1,2) = 6 \) while \( f(3,4) = 9 \). Now let \( C \) be any continuous curve that joins \( P \) to \( Q \). The path generated by a line parallel to the z-axis running along \( C \) gives us a cylinder. (Recall that mathematically a cylinder is obtained by tracing along any curve in the plane with a line perpendicular to the curve), and this cylinder intersects the surface \( z = f(x, y) \) to form an unbroken (space) curve. All we are now saying is that since

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this space curve is unbroken, and it passes through the heights 
6 and 9, it must pass through every height between 6 and 9. 
Stated in terms of planes, we are saying that our space curve 
originates in the plane $z = 6$ and terminates in the plane 
$z = 9$. Since the curve is unbroken, it must intersect every 
plane $z = k$ where $6 < k < 9$.

Thus, still using our particular example, since $6 < 8 < 9$, 
there is a point $P_1$ on the continuous curve $C_1$ which connects 
$(1,2)$ and $(3,4)$ for which $f(P_1) = 8$.

Notice that $C_1$ was any curve that connected $(1,2)$ and $(3,4)$. 
The point is that there are infinitely many such continuous 
curves which have no points other than $(1,2)$ and $(3,4)$ in common. 
For each of the curves $C_n$, there is a point $P_n$ for which $f(P_n) = 8$. 
Since the curves intersect only at $(1,2)$ and $(3,4)$ the point 
$P_1$ on $C_1$ cannot be the same as the point $P_j$ on $C_j$ (no $P_n$ can 
be either $(1,2)$ or $(3,4)$ because $f(P_n) = 8$ while $f(1,2) = 6$ 
and $f(3,4) = 9$). Thus, we have infinitely many different points 
$P_1', P_2', ..., P_n', ....$ for which $f(P_1') = f(P_2') = ... = f(P_n') = ... = 8$.

An attempt to summarize these results pictorially is made below:
1. $C_n$ is a continuous curve in the xy-plane connecting P(1,2) and Q(3,4).

2. The (right) cylinder along $C_n$ intersects the surface $z=f(x,y)$ along the space curve $P'Q'$.

3. $P'$ is in the plane $z=6$ since its coordinates are (1,2,6). $Q'$ is in $z=9$ since its coordinates are (3,4,9).

4. The plane $z=8$ intersects $P'Q'$ at, at least, one point $Q_n$ since $P'Q'$ is a continuous curve.

5. The projection of $Q_n$ into the xy-plane yields $P_n$ on $C_n$ whereupon $f(P_n) = 8$.

(Figure 1)

The important thing in Figure 1 is that $C_n$ was any continuous curve that joins P to Q. There are infinitely many such curves with no points of intersection other than P and Q. In Figure 2 four such curves $C_1$, $C_2$, $C_3$, $C_4$ are drawn. By the construction in Figure 1, we may obtain $P_1$ on $C_1$, $P_2$ on $C_2$, $P_3$ on $C_3$, $P_4$ on $C_4$ such that $f(P_n) = 8$ for every $n = 1,2,3,4$. 
The point we now wish to make actually begins with the solution of this exercise. We have now shown that if \( f: \mathbb{R}^2 \to \mathbb{R} \) is any continuous function, then it is impossible for \( f \) to be 1-1. In fact, in the particular illustration we used, there were infinitely many points in the domain of \( f \) at which \( f \) was \( 8 \).

What this exercise shows is that if the mapping from \( n \)-dimensional space (with \( n > 1 \)) into 1-dimensional space is continuous, then \( f \) cannot be 1-1. Restated more formally:

If \( n > 1 \) and \( f: \mathbb{R}^n \to \mathbb{R} \) is continuous then \( f \) is not 1-1, and, in fact, there can be infinitely many elements in \( \mathbb{R}^n \) which have the same image in \( \mathbb{R} \).

(Notice that our argument holds if the domain of \( f \) is any subset of \( \mathbb{R}^n \). That is, all that our argument required was that \( f(P) \neq f(Q) \) no matter how close to each other \( P \) and \( Q \) might be.)

Thus, we should begin to feel that while much of the theory for functions of \( n \)-tuples is a generalization of the results for the 1-dimensional case, there are enough new "wrinkles" when \( n > 1 \) to
remind us to be on guard. Among other things, there are infinitely many continuous functions $f:E^1 \rightarrow E$ which are 1-1 (namely those whose graphs are either monotonically rising or monotonically falling).

By now, we hope that it is virtually self-evident to you that $n=4$ was chosen simply for concreteness and that our approach extends verbatim to any value of $n$.

To prove that "=" is an equivalence relation we must show that

1. $a = a$ for each $a \in E^4$
2. $a = b \rightarrow b = a$, $a$ and $b$ in $E^4$
3. $a = b$ and $b = c \rightarrow a = c$

As for (1), we have $a = (a_1, a_2, a_3, a_4)$ where $a_1, a_2, a_3, a_4$ are all real numbers. By the properties of equality for real numbers, we know that

$a_1 = a_1$, $a_2 = a_2$, $a_3 = a_3$, and $a_4 = a_4$.

Hence, by definition of vector (n-tuple) equality, we have

$(a_1, a_2, a_3, a_4) = (a_1, a_2, a_3, a_4)$

or

$a = a$

which establishes the validity of (1).

(Notice how (1) was established for $E^4$ by virtue of the corresponding property of equality for real numbers).
As for (2), let \( a = (a_1, a_2, a_3, a_4) \) and \( b = (b_1, b_2, b_3, b_4) \). Then, by definition, \( a = b \) means

\[
\begin{align*}
    a_1 &= b_1, & a_2 &= b_2, & a_3 &= b_3, & a_4 &= b_4
\end{align*}
\]

But, for real numbers, equality is symmetric. Hence, we may conclude that

\[
\begin{align*}
    b_1 &= a_1, & b_2 &= a_2, & b_3 &= a_3, & b_4 &= a_4
\end{align*}
\]

whence, by definition,

\[(b_1, b_2, b_3, b_4) = (a_1, a_2, a_3, a_4)\]

or

\[b = a\]

which establishes the validity of (2).

Finally, to establish the validity of (3), we let

\[a = (a_1, a_2, a_3, a_4), \quad b = (b_1, b_2, b_3, b_4) \quad \text{and} \quad c = (c_1, c_2, c_3, c_4).\]

Then, from \( a = b \) we conclude that

\[
\begin{align*}
    a_1 &= b_1, & a_2 &= b_2, & a_3 &= b_3 \quad \text{and} \quad a_4 &= b_4, \quad (1)
\end{align*}
\]

while from \( b = c \) we conclude that

\[
\begin{align*}
    b_1 &= c_1, & b_2 &= c_2, & b_3 &= c_3 \quad \text{and} \quad b_4 &= c_4 \quad (2)
\end{align*}
\]

Since for real numbers, \( a = b \) and \( b = c \cdot a = c \), we may conclude from (1) and (2) that

\[
\begin{align*}
    a_1 &= c_1, & a_2 &= c_2, & a_3 &= c_3 \quad \text{and} \quad a_4 &= c_4.
\end{align*}
\]
Hence

\((a_1, a_2, a_3, a_4) = (c_1, c_2, c_3, c_4)\)

or

\(a = c\)

Notice how this proof augments our remark in the supplementary notes that the fact that \(=\) is an equivalence relation on \(E^4\) is induced by the fact that vectors may be viewed as \(n\)-tuples, each of whose components is real, and equality is an equivalence relation on the real numbers.

3.1.11

(Note: The technique used in this specific exercise is the one used whenever we decide to view vectors in terms of their components).

We have, by definition of the arithmetic of \(E^4\), that

\[ b + c = (b_1, b_2, b_3, b_4) + (c_1, c_2, c_3, c_4) \]

\[ = (b_1 + c_1, b_2 + c_2, b_3 + c_3, b_4 + c_4) \]

Hence, by our definition of dot product

\[ a \cdot (b + c) = (a_1, a_2, a_3, a_4) \cdot (b_1 + c_1, b_2 + c_2, b_3 + c_3, b_4 + c_4) \]

\[ = a_1(b_1 + c_1) + a_2(b_2 + c_2) + a_3(b_3 + c_3) + a_4(b_4 + c_4) \quad (1) \]

Since the \(a_k\)'s, \(b_k\)'s and \(c_k\)'s \((k = 1, 2, 3, 4)\) are numbers, we know that \((1)\) may be rewritten as

\[ a_1b_1 + a_1c_1 + a_2b_2 + a_2c_2 + a_3b_3 + a_3c_3 + a_4b_4 + a_4c_4 \]
which, in turn, is

\[(a_1b_1 + a_2b_2 + a_3b_3 + a_4b_4) + (a_1c_1 + a_2c_2 + a_3c_3 + a_4c_4) = a \cdot b + a \cdot c\]

which proves the required result.

Again, notice how we effectively reduce the study of \(E^4\) to four separate studies of \(E^1\) (i.e., the four components of any vector [n-tuple] in \(E^4\)). That is, the vector result is induced by the corresponding scalar result.