**HERBERT GROSS:** Hi. I hope enough time has elapsed since our last lecture so that you feel at home doing arithmetic of complex numbers. Because you see now, what we'd like to do today is the next phase that leads into our calculus study of complex numbers. Namely, what we would like to do is study functions of a complex variable. And again, in terms of a function machine, I'm visualizing the input of my machine being a complex number and the output being a complex number.

By the way, I should mention that there are situations, and we'll cover these in the homework exercises, where one sometimes likes the input to be a real number and the output a complex number or the input a complex number and the output a real number. But for reasons that I hope will become apparent in a moment, we're going to focus our attention on the case where both the input and the output are complex numbers.

Noticing that if we do that, especially from a geometrical point of view, a vector point of view, our input is a two tuple, say the complex number x comma y or x plus i-y as represented by the two tuple x comma y. The output, w equals f of z, is u plus i-v, say, or as a two tuple, u comma v. The idea being that graphically we can view a function that maps the complex numbers into the complex numbers as a mapping from the xy plane into the uv plane where the complex number z is identified with the pair x comma y, the complex number w is identified with the ordered pair u comma v. The xy plane is the domain of our function. The uv plane is the range of our function.

By way of example, suppose I take the function f of z equals z squared. Recalling that z is x plus i-y, if I square this I get as the real part x squared minus y squared. The imaginary part is 2xy. So as a two tuple type mapping, f maps x comma y into x squared minus y squared comma 2xy. In other words, u is x squared minus y squared. v is 2xy.

So in summary, notice that the complex value function of a complex variable-- in other words, we're mapping complex inputs into complex outputs. With that particular mapping, f of z equals z squared is equivalent to the-- and this is important-- to the real system of equations u equals x squared minus y squared, v equals 2xy.

You see, this is a system of two real equations and two unknowns. And what I'm saying is that I cannot distinguish between this problem and this problem. In other words, somehow or other, studying complex valued functions is going to help me replace or to view from a different vantage point the study of systems of real functions. But again, more about that as we proceed.

Anyway, leaving further drill-- further drill? We haven't done any drill. Leaving drill to the exercises because I think conceptually this is all old hat. And why do I say it's old hat? Well, heck, we've spent a whole block of material working on these kinds of mappings. Also, geometrically, we're viewing the complex numbers as being vectors so that in another manner of speaking, complex valued functions of a complex variable are equivalent to what? Functions that map two dimensional space into two dimensional space. And we talked about that in our treatment of vector functions.

So whichever way we look at this thing, I think that the drill is fairly straightforward. Let's go on. The next thing after we have functions. See, ultimately, we want to get to derivatives and integrals. And the one concept that we need for a derivative, after we know the theory of functions, is the concept of a limit. So we want limits next.

And what are we going to do letting C denote the complex number just for purposes of identification. Suppose we have a mapping from C into C. Suppose a is in C. In other words, a is some complex number. We define the limit of f of z as z approaches a to [INAUDIBLE].

And I'm hoping that by this time you're way ahead of me because we've done this many times. What am I going to do here? I am going to, word for word, mimic the real definition that was the case for functions of a real variable, or if you wish for vector functions. See this is complex. This is complex. I can think of this as a vector, this as a vector. Same definition that we had before, namely given epsilon greater than 0 there exists a delta greater than 0 such that when the absolute value of z minus a is greater than 0 but less than delta, the absolute value of f of z minus L is less than epsilon, noticing that even though z, a, f of z, and L are complex, magnitudes are real.

So all of these symbols make sense. And do they capture intuitively what we'd like them to mean? Again, by way of a quick pictorial review, the answer is exactly yes. Namely if we think of wanting to be with an epsilon of L, we simply now in two dimensions take a circle of radius epsilon centered et L. And what we're saying is to be with an epsilon of L, we can find a circle of radius delta centered at a such that whenever z is in this circle, f of z will be in this circle, which is exactly the intuitive feeling that limit is supposed to convey.

Because structurally the definition of limit is the same now as it was in the real case and in the real case we only used properties of a definition to prove our theorems, it means that the usual limit theorems will hold. And in particular, by the way, if we decide to write f of z as a real plus an imaginary part, in other words f of z is what? It's u plus i-v where u and v are functions of x and y.

The complex number L has a real part and an imaginary part. L is L1 plus iL2 where these are real, you see. So this is complex. These are real. And the complex number a can be written in the form a-1 plus ia-2 where a1 and a2 are real.

And essentially what we're saying is that if the limit as z approaches a, f of z is L, it must be that the real part of f of z approaches the real part of L and the imaginary part of f of z approaches the imaginary part of L. In other words, this condition here is equivalent to a system of two limits in real variables, namely the limit of u of xy as xy approaches a1 and a2 must be the real number L1. And the limit of v of xy as xy approaches the real ordered pair a1, a2 must be L2.

And again what this means is that we can often view systems of limit problems as being the limit of a complex variable function. Again, that's all we're saying from that point of view. I simply want to rush through this because this is old hat. There is really nothing new theoretically here. All I want you to see is that the same structure that held in our study for real valued functions holds for complex valued functions. But in particular, when we map complex numbers into complex numbers, there are many real interpretations identified with the pair of real mappings u of some function of x and y and v of some function of x and y.

At any rate, with that as background, we can immediately launch into the definition of a derivative. Now we may not be able to immediately launch into what does a derivative mean. This is one of the beauties, you see, of mathematical structure.

One of the beauties of mathematical structure is is that when I'm mimicking a structure, all I have to do is copy it. That's what mimic means, in fact. The beauty isn't that you can copy it. The beauty is is that since all of the powerful results of the thing that you're copying came logically-- in other words, they were valid conclusions based on the structure you started with-- the new structure that you got by mimicking the old one will inherit the same structure , provided, of course, that in mimicking the resulting expression still makes sense.

Without beating around the bush, all I'm saying is why don't we copy as our definition of f prime of z sub 0 where z sub 0 is some complex number, why don't we mimic what we would have done for f prime as x sub 0, only replacing the x's by the z's.

And if we do that, remembering what our definition for derivative was in the case of a real variable, we simply define f prime of z0 to be the limit as delta z approaches 0, f of z0 plus delta z minus f of z0, all over delta z. The questions, of course, that we raise are, first of all, does this make sense.

Well, remember f is defined for complex numbers. And an f of a complex number is a complex number. So this is the difference of two complex numbers. The difference of two complex numbers is a complex number. We're dividing that difference by another complex number, which by the way can't be 0. Remember, delta z is not 0. It approaches 0. It means that delta z is greater than 0 but less than delta, et cetera, little delta, the same as in the epsilon-delta definition.

This says what? The delta z gets arbitrarily close to 0 but it's never equal to 0. Consequently, this is the quotient of two complex numbers. The denominator is not 0. Consequently, as we saw in the last lecture, this is again a complex number. And this makes sense to compute this limit. And we'll actually do this in a problem before we're through today. And we'll also have a lot of homework problems on this.

But to try to give you some feeling as to what this thing means geometrically, think of this in terms of u and v again. In other words, think of mapping the xy plane into the uv plane. If w is f of z where f of z is what, u plus i-v, then f prime of z0 is just the symbol dw dz whereby the dw dz we mean the limit as delta z approaches 0, delta w over delta z.

And by the way, to help you see that hopefully a little bit more clearly, if this is the w, delta w involves a change in u and a change in v. So delta w would be delta u plus i-delta v. Delta z, by definition, is delta x plus i-delta y. So computing the derivative involves computing this limit. And notice that with the exception of the i appearing here, everything else involves real numbers.

The very, very crucial thing to notice is that not only must this limit exist. But it must exist independently of the direction in which delta z approaches 0. This is much stronger than the corresponding notion of a directional derivative when we were talking about real valued functions of several real variables.

You see, for a directional derivative to exist in every direction all you require is that in each direction the limit exists, but that the limit could be different in two different directions. Notice that in our definition, this says what? That this particular limit must exist no matter what direction delta z approaches 0 in.

Now to help you see this geometrically, and for heaven's sakes I'm not the greatest guy at visualizing things geometrically myself, so if I either confuse you or else you're not turned on by this, ignore it. Because we don't need this geometric interpretation until next lecture.

But for those of you who might be interested in pursuing this, here's what we're saying. Notice that f prime of z0 is a complex number. It's in the uv plane. Complex numbers are like vectors. Consequently, I can visualize the complex number as a directed length, an arrow.

If I take f prime of z0 let me assume it originates at w0 where w0 is the image of z0 under f. So as a vector, f prime of z0 is this. Now here's what we're saying. Pick any point z1 near z0. Any point at all, z1. It maps into some point w1.

The directed length from z0 to z1 is the vector of the complex number delta z. The directed distance from w0 to w1 is the vector, the complex number, delta w. In general, one would expect that when you divide delta w by delta z, what you get should depend very strongly on what z1 is. See, different values of z1 would have led to different values if w1.

The fact that the limit of delta w over delta z is f prime of z0 means that if z1 is very close to z0, no matter where you pick it, the ratio delta w by delta z is essentially a constant. In fact, what constant is it? It's essentially the vector f prime of z0, the complex number f prime of z0, contrary to what you might suspect that in general different z's will lead to such drastically different w's that the ratio delta w by delta z is bound to change. What we're saying is if a function is differentiable, that ratio doesn't change. And it's always equal to f prime of z0.

Now the question comes up is it's very, very strong condition to insist not only that this limit exists in every direction but be the same in all directions. It's possible that this imposes some very strong conditions on delta u and delta v. And what we're going to do is to say lookit, if this must be the same for all directions, let's pick two particularly attractive directions to look at.

What directions do we like when we deal with real functions of two real variables? What we like is, in that case we're dealing with what? A function of x and y, in which case the usual derivatives involve the partials with respect to x and the partials with respect to y. That means, in one case, holding delta x equal to 0. You're holding x constant so delta x is 0. The The other case, you hold y constant. So delta y is 0.

Let's look at these two cases specially. In other words, let's look at this first of all in the special case that delta y is identically 0. You see, if delta y is identically 0, delta z is just delta x in that case. And so this drops out. And the problem would be to compute the limit as delta x approaches 0 delta u plus i-delta v over delta x.

And saying that more slowly so that you can see it better, if I let delta y be identically 0, it means that I'm allowing delta z to approach 0 horizontally. In other words, I'm allowing the approach to be parallel to the real axis. That's how we're approaching z0 now.

In that case, since delta y is 0, delta z is just delta x. This is the limit as delta x approaches 0 delta u plus i-delta v over delta x. That means I can break this down and write it as delta u over delta x plus i-delta v over delta x.

The limit of a sum is the sum of the limits, et cetera. I take the limits here recalling that the limit of delta u over delta x as delta x approaches 0 holding y constant-- see delta y is 0. Here's what I mean by the partial with respect to x. Similarly, the limit of delta v over delta x as delta x approaches 0 and delta y is 0-- in other words y is constant-- is what we mean by delta v over delta x.

And consequently, what we're saying is if f prime of z0 exists, it must be the partial of u with respect to x plus i times the partial of v with respect to x. And I'm computing this at z0, which corresponds to the point in the xy plane, say x0, y0.

On the other hand, I know that f prime of z0 has to exist and be the same if delta x is held to be identically 0. Holding delta x0-- in other words, holding x constant-- means that you're now allowing z to approach z0. Delta z is approaching 0 parallel to the imaginary axis, the y-axis.

Now in this case, what happens? If delta x is identically 0, this term drops out. Delta z is just i-delta y. Since i is a constant different from 0, the only way that i-delta y can approach 0 is if delta y approaches 0. Consequently, with delta x0, this reads the limit as delta y approaches 0 delta u plus i-delta v over. i-delta y.

And if I now divide each term in here by i-delta y, you see what's going to happen. I'm going to get a delta u over i-delta y here. And the i's will cancel here. So that f prime of z0 will turn out to be the limit as delta y approaches 0 delta u over i-delta y plus delta v over delta y. Again taking to the limit as delta y approaches 0, remembering that x is being held constant, and remembering what the definition for the partials of u and v with respect to y are. This limit simply turns out to be this.

I would like to write this in the standard form-- something real plus something real times i, not 1 over i. And so I use that usual thing of the complex conjugate. I multiply numerator and denominator here by minus i. That gives me my new numerator is minus i. My new denominator is minus i times i, which is minus minus 1, or plus 1. So that f prime of z0 is the partial of u with respect to y minus i partial of u with respect to y. Or if you wish, it's plus i times minus partial u with respect to y.

At any rate, without making a big issue over this, since these two must be the same and since the only way that two complex numbers can be equal is if they're equal component by component. In other words, the real parts must be equal. And the imaginary parts must be equal . I compare the real parts here and the imaginary parts and conclude that they must be equal.

So in summary if f is equal to u plus i-v and f is differentiable-- and by the way a very common word that's used when a complex valued function of a complex variable is differentiable is that we say it's analytic. If f equals u plus i-v is analytic, then the partial of u with respect to x is v sub y. In other words, the partial of u with respect the y. And the partial of u with respect to y must be minus the partial of u with respect to x. You see, this equals this. And this must line up with this.

By the way, notice that we arrived at this result using only the information of two special directions. We had delta x0 and delta y0. So all this proves is is that if the function is differentiable this must be true. It happens that the converse is also true. And we're going to prove that as an exercise in the unit.

Intuitively, to respect why this is true, it's the same reason as why the directional derivative is determined by the partials with respect to x and y. In other words, if the partials exist and are continuous, knowing what happens in the x and y direction is enough to tell you what's happening in every direction. And we'll give you the rigorous details as a exercise later on.

But to put this from a different perspective, if either u sub x is not equal to v sub y or u sub y is not equal to minus v sub x, then u plus i-v cannot be an analytic function. And the easiest way to talk about this is in terms of examples. Let's look at this. Let's take three examples, but hopefully one at a time.

The first example is f of z equals z squared. And by the way, before I go any further, notice that if you had not thought of any subtleties here and I said what is f prime of z, the odds are overwhelming that you would have said 2z right away. I want to show you that that really is the case. And I want to do it in two different ways.

One is, as we've seen before, z squared can be written as x squared minus y squared plus i times 2xy. So that corresponds to u equals x squared minus y squared, v equals 2xy from which you see we can compute u sub x, u sub y, v sub x, and v sub y. And what we see right away is that u sub x does equal v sub y, u sub y does equal minus v sub x. So in particular, these conditions are obeyed.

This is written up in the text by the way, the proof of what we've just done. The conditions are called the Cauchy-Riemann conditions. That's also mentioned in the text. So I won't take the time to write that now. But the idea is that what? Because the Cauchy-Riemann conditions are obeyed, it means that this function should be differentiable.

By the way, knowing that it should be differentiable is not the same as saying that you know what the derivative is. Let's now take the alternative approach that somehow or other we don't look at u and v but work directly from our basic definition. After all, we define what f prime of z meant in terms of f of z and f of z plus delta z. Consequently, this should have meaning regardless of any interpretation in terms of u's and v's.

Notice that if I write this, by definition of this, this quotient f of z0 plus delta z minus f of z0 over delta z is just z0 plus delta z squared minus z0 squared over delta z. The binomial theorem works for complex numbers the same way as it does for real numbers. In fact, that was what motivated our definition for multiplying complex numbers. if you recall, our lecture of last time was to mimic what happens in the real case.

This, the same as before, comes out to be what? z0 squared, which kills off this minus z0 squared plus 2z0 delta z plus delta z square, all over delta z. I'm assuming I'm now going to take the limit as delta z approaches 0. That means, in particular , delta z is not 0. Because delta z is not 0, I can divide through by it. That gives me 2z0 plus delta z.

And now notice if I let delta z approach 0, z0 is a fixed number. That doesn't change as delta z approaches 0. On the other hand, delta z approaches 0 as delta z approaches 0, that's a truism. Delta z approaches 0, no matter what the direction of approach is. This is delta z goes to 0, it goes to 0.

This term does drop out. And you wind up with the anticipated result that f prime of z0 is 2z0. Notice again, structurally, that in the case where there is an analogy between the complex variable and the real variable, the steps are verbatim because the definitions were mimicked to be verbatim, that you cannot tell the difference between the proof that f prime of z is 2z in this case with the proof that f prime of x was to 2x when f of x was equal to x squared in the real case.

Or as another example, one would expect that the derivative of f of z, if f of z is 1 over z, should be minus 1 over z squared. Except that when z is 0, this shouldn't be differentiable. I leave it again as a voluntary exercise for you to apply this definition to 1 over z, 1 over z plus delta z, et cetera, and show that step by step you get the same result as in the corresponding problem involving 1 over x.

To show you how this works in terms of u's and v's, just for the drill, if z is x plus i-y, 1 over z is 1 over x plus i-y. Multiplying numerator and denominator by the complex conjugate x minus i-y here, my denominator becomes x squared plus y squared. My numerator is x minus i-y.

That's the same as x over x squared plus y squared plus i times minus y over x squared plus y squared. This is the u part. This is the v part. u equals x over x squared plus y squared. v equals minus y over x squared plus y squared. I again leave it as an exercise for you to show that the Cauchy-Riemann conditions are obeyed. In other words, that the partial of u with respect to x equals the partial of u with respect to y. The partial of u with respect to y is minus the partial of u with respect to x.

And the only place that this isn't true turns out to be when this denominator is 0. The only time this denominator can be 0 is when x and y are both 0. In other words, the Cauchy-Riemann conditions here are obeyed except when x equals y equals 0.

And that, of course, implies what? That z is 0, the real and the imaginary parts of 0. And what we're saying is that 1 over z is analytic except when z is 0. And in fact, the derivative is minus 1 over z squared.

For a third example, I would like to pick one that we can't-- that has no analog in the real case. I want to pick a genuine complex valued function of a complex variable. And I'm going to pick a very simple one. And the reason I picked a simple one is twofold. One is the computations are easy to do. The result is easy to interpret geometrically.

And thirdly and most importantly, I mentioned before that it is not easy for a complex valued function to have a derivative. A directional derivative must exist in every direction and be the same. That's a very stringent condition. So I picked this third example to show you how a relatively simple complex valued function of a complex variable can have no derivative.

And the simple function that I have chosen is the one which maps the complex conjugate, the number into its complex conjugate. The complex number z is mapped by f into f bar. In terms of two tuples, if the input of the f machine is x comma y, the output will be x comma minus y. A very simple thing, very straightforward, easy function machine to work with here.

Notice in this case that u is x. v is minus y. The partial of u with respect to x is 1. The partial of u with respect to y is minus 1. 1 is not equal to minus 1. Consequently, u sub x is not equal to v sub y. That means that the Cauchy-Riemann conditions are not obeyed. By what we just saw a few minutes ago, that in turn means that this function does not have a derivative.

Well, that's nice to know the function doesn't have a derivative. It would be even nicer if we could see why it didn't. And the best way to do this is, again, go back to the definition. What we're going to ultimately do to take a derivative is we're going to take delta w where w is z bar. Delta w over delta z and take the limit as delta z approaches 0 along thee various directions.

Well, delta w is just delta x minus i-delta y. Delta z is delta x plus i-delta y. To keep in mind that I'm going to move in a particular direction in the xy plane to approach z0, I'm going to it up the slope over here. Namely in this expression, I'm going to assume that delta x is not 0. And what I'm going to do is divide through a numerator and denominator by delta x. That gives me 1 minus i times delta y over delta x over 1 plus i-delta y over delta x.

And as I approach the point z0 in the xy plane, delta y over delta x approaches the slope of the curve at z0, which we'll call dy dx. And I wind up with this.

Now to simplify this, let me assume that I approach z0 along some straight line. Let me pick a fixed straight line. I'll pick a straight line which goes through z0 and has slope m. So in that case, dy dx is the same as delta x is m for all points on this particular line.

So what is dy dx in this case? It's m. Consequently, what is delta w over delta z in this case? And by the way, notice what I'm saying here. What I'm saying is this is this line segment. z0 maps into w0. z1 maps into w1, sufficiently close to z0 I can assume that the image of this straight line is a straight line that joins w0 to w1.

And the ratio that I want is this length divided by this length, where by dividing lengths I mean the usual complex variable interpretation, dividing magnitudes and subtracting arguments. At any rate, what we're saying is that delta w over delta z in this case is just 1 minus i-m over 1 plus i-m. And notice that as delta z approaches 0 in this direction, this limit exists. In fact, there's no variable in here.

It's 1 minus i-m over 1 plus i-m. But notice that even though that limit exists, the value depends on m. If I change m, I change this ratio. And because that limit depends on m, it means that even though the directional derivative exists different directions will give me-- see, if I come in along this line I can compute this ratio of delta w over delta z. Will I get in that case? 1 minus i-m 1 over 1 plus i-m 1.

That limit will exist. But it will be unequal to this one. You see, the derivative exists in each direction. But it's not the same in each direction. That's how subtle this is. I will leave the remaining exercises-- not the remaining exercises but exercises to hit home at the remaining fine points.

What I wanted to do in closing today's lesson was to show you one more tremendous connection with complex variables, or between complex variables and real problems in the physical world. You may recall from some of our previous exercises in an earlier block that we say that u of xy satisfies Laplace's equation if the second partial of u with respect to x plus the second partial of u with respect to y is 0. In fact, if u denotes temperature distribution at the point-- the temperature at the point x comma y, this equation is what's known as the steady state equation. We're not going to worry about the physical ramifications here.

What is important is that to solve this partial differential equation, which comes up in the physical world, this is a real partial differential equation meaning real in two senses. It comes up in the real world. And also in the sense that u is a real valued function of x and y.

Now the interesting point is this. That if u plus i-v is analytic, what do we know about u plus i-v being analytic? It means that the partial of u with respect to x is v sub y. The partial of u with respect to y is minus the partial of v with respect to x. If I differentiate both sides here with respect to x and both sides here with respect to y, I get u sub xx equals v sub yz. u sub yy is minus v sub xy.

Assuming continuity so that these are equal, this says by addition that if I add these two, I get the second partial of u with respect to x plus the second partial of u with respect to y is 0. A similar argument would have shown that the same is true for the imaginary part, that the second partial of u with respect to x plus the second partial of u with respect to y also will be 0.

In other words, the real and the imaginary parts which by themselves are real valued functions of two real variables, the real and imaginary parts each satisfy Laplace's equation. The converse, by the way, is also true. I'll mention that in a moment.

But for the time being, let me close today's lesson with an example. We already know that f of z equals z squared is analytic. It's differentiable. The real part of z squared u is x squared minus y squared. The imaginary part v is 2xy.

What that tells me without even checking is that if I were to form u sub xx plus u sub yy or v sub xx plus v sub yy, those would be 0. Because of the fact that they satisfy the Cauchy-Riemann conditions, which is what we'll talk about in more detail during the exercises. And we'll bring this up more in the next lecture.

But for the time being, what I wanted to close on aside from the calculus was the application. I want you to see that solving LaPlace's equation is a very important real world problem. The interesting point is that the solutions of the LaPlace's equations have to be the real and imaginary parts of analytic functions. And there aren't that many analytic functions. And consequently by knowing analytic functions, we have a good hold on what solutions of Laplace's equation in the real world look like.

Well, I think that's enough for one session. Work on the exercises. And next time we're going to pound out some other interesting ramifications of what it means for a complex valued function to be differentiable. At any rate, then, until next time, goodbye.

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