Unit 3: Additional Comments on Dimension

1. Overview

In most respects this Unit could have been included as a subtopic of the previous one, but we have elected to include it as a separate unit in order to give you another chance for grasping the "big picture".

2. Lecture 3.030

(a) Constructing Bases

Review

1. Let \{v_1, \ldots, v_n\} be a basis for \(V\) in \(V^n\).
2. If \(n\) is a basis for \(V\), then any \(n\) elements of \(V\) are linearly independent.
3. If \(n\) is a basis for \(V\), then any \(n\) elements of \(V\) must span \(V\).
4. If \(n\) is a basis for \(V\), then any \(n\) elements of \(V\) cannot span \(V\).
5. Every basis for \(V\) has \(n\) elements.
6. Any set of \(n\) elements of \(V\) which span \(V\) is a basis for \(V\).
7. \(\dim V = n\) and \(\{v_1, \ldots, v_n\}\) is a particular basis for \(V\) if and only if every \(v\) in \(V\) has a unique representation as \(\sum_{i=1}^{n} c_i v_i\), where \(c_i\) are unique coefficients.

Relative to \(\{v_1, \ldots, v_n\}\) we may write \(v = (c_1, \ldots, c_n)\), and we then write \(v = [c_1, \ldots, c_n]\).

Example

Let \(\dim V = 4\) and suppose \(V = \mathbb{R}^4\). Suppose \(\beta = (1, 0, 0, 0), \gamma = (0, 1, 1, 0), \delta = (0, 0, 1, 1), \epsilon = (0, 0, 0, 1)\) are a basis for \(V\).

Describe \(W = \text{Span}(\beta, \gamma, \delta, \epsilon)\). Here, \(W = \mathbb{R}^4\).

Then \(W = \mathbb{R}^4\).

If \(w = (1, 0, 0, 1)\), then \(w = (1, 0, 0, 0) + (0, 0, 0, 1) = \beta + \epsilon\) and \(w = (1, 0, 0, 0) + (0, 0, 1, 1) = \gamma + \delta\).

If \(w = (0, 1, 1, 0)\), then \(w = (0, 0, 0, 1) + (0, 0, 1, 0) = \epsilon + \delta\).

If \(w = (1, 0, 0, 0)\), then \(w = (1, 0, 0, 0) + (0, 0, 0, 0) = \beta + \gamma\).

Hence, \(W = \mathbb{R}^4\).

Matrix Technique

Compute the \(5 \times 4\) matrix \([\begin{array}{cccc} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \\ 13 & 14 & 15 & 16 \\ 17 & 18 & 19 & 20 \end{array}]\).

Then, \(\beta = (1, 0, 0, 0), \gamma = (0, 1, 0, 0), \delta = (0, 0, 1, 0), \epsilon = (0, 0, 0, 1)\) are a basis for \(V\).

Compute the \(4 \times 4\) matrix \([\begin{array}{cccc} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \\ 13 & 14 & 15 & 16 \end{array}]\).

Then, \(\beta = (1, 0, 0, 0), \gamma = (0, 1, 0, 0), \delta = (0, 0, 1, 0), \epsilon = (0, 0, 0, 1)\) are a basis for \(V\).

Compute the \(3 \times 4\) matrix \([\begin{array}{cccc} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \end{array}]\).

Then, \(\beta = (1, 0, 0, 0), \gamma = (0, 1, 0, 0), \delta = (0, 0, 1, 0), \epsilon = (0, 0, 0, 1)\) are a basis for \(V\).
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c. \[
\begin{bmatrix}
1 & 2 & 3 & -1 & 0 \\
0 & 1 & 0 & -2 & 1 \\
0 & 0 & 5 & -4 & 1 \\
0 & 0 & 1 & 2 & -1
\end{bmatrix}\]

\[
\begin{align*}
\begin{pmatrix}
1 & 2 & 3 & -1 & 0 \\
0 & 1 & 0 & -2 & 1 \\
0 & 0 & 5 & -4 & 1 \\
0 & 0 & 1 & 2 & -1
\end{pmatrix}
&= \begin{pmatrix}
1 & 2 & 3 & -1 & 0 \\
0 & 1 & 0 & -2 & 1 \\
0 & 0 & 5 & -4 & 1 \\
0 & 0 & 1 & 2 & -1
\end{pmatrix}
\begin{pmatrix}
1 & 2 & 3 & -1 & 0 \\
0 & 1 & 0 & -2 & 1 \\
0 & 0 & 5 & -4 & 1 \\
0 & 0 & 1 & 2 & -1
\end{pmatrix}^{-1}
\end{align*}
\]

d. The matrix equation is:

\[
\begin{bmatrix}
1 & 2 & 3 & -1 & 0 \\
0 & 1 & 0 & -2 & 1 \\
0 & 0 & 5 & -4 & 1 \\
0 & 0 & 1 & 2 & -1
\end{bmatrix}
\begin{pmatrix}
1 \\
2 \\
3 \\
4
\end{pmatrix}
= \begin{pmatrix}
3 \\
7 \\
9 \\
11
\end{pmatrix}
\]

Summary of Example:
- \(w = (1, 2, 3, -1, 0) = (a, b, c)\)
- \(\text{dim } w = 3\)
- \(w = \{(1, 2, 3) \mid (1, 2, 3) = (a, b, c)\}\)
- \(a = 1, b = 2, c = 3\)
- \(\begin{pmatrix}
1 & 2 & 3 & -1 & 0 \\
0 & 1 & 0 & -2 & 1 \\
0 & 0 & 5 & -4 & 1 \\
0 & 0 & 1 & 2 & -1
\end{pmatrix}
\begin{pmatrix}
1 \\
2 \\
3 \\
4
\end{pmatrix}
= \begin{pmatrix}
3 \\
7 \\
9 \\
11
\end{pmatrix}
\]

3.3.2
3. Exercises:

3.3.1 (L)

Let dim \( V = 4 \) and assume that \([u_1, u_2, u_3, u_4]\) is the coordinate system being used for denoting the elements of \( V \) as 4-tuples. Let \( W \) be the subspace of \( V \) generated by \( \alpha_1 = (1, 1, 3, 4) \), \( \alpha_2 = (2, 3, 7, 9) \), \( \alpha_3 = (3, -2, 4, 7) \), \( \alpha_4 = (4, -5, 3, 7) \), and \( \alpha_5 = (4, 5, 14, 9) \).

a. Find the dimension of \( W \).

b. Express \( x_4 \) as a linear combination of \( x_1, x_2, \) and \( x_3 \) if it is known that \((x_1, x_2, x_3, x_4) \in W\).

c. Find vectors \( \beta_1, \beta_2, \beta_3 \in W \) such that \((x_1, x_2, x_3, x_4) \in W \) \(\iff\) \((x_1, x_2, x_3, x_4) = x_1\beta_1 + x_2\beta_2 + x_3\beta_3\). Then express \( \alpha_1, \alpha_2, \) and \( \alpha_5 \) as linear combinations of \( \beta_1, \beta_2, \) and \( \beta_3 \).

3.3.2

Let \( V = [u_1, u_2, u_3] \) and define \( \alpha_1, \alpha_2, \alpha_3 \) by \( \alpha_1 = (5, 2, 7) \), \( \alpha_2 = (-3, 4, 1) \), and \( \alpha_3 = (1, -2, -3) \). Let \( W = S(\alpha_1, \alpha_2, \alpha_3) \).

a. Show that \( \dim W = 2 \).

b. Find a linear combination of \( \alpha_1, \alpha_2, \alpha_3 \) which is zero even though no coefficients are zero.

c. Show that \( \alpha_1 \) may be written in infinitely many different ways as a linear combination of \( \alpha_1, \alpha_2, \) and \( \alpha_3 \).

3.3.3 (L)

Show that \( W = \{ f : f''(x) - 4f(x) = 0 \} \) is a subspace of the space of continuous functions.

3.3.4 (L)

Let \( V = [u_1, u_2, u_3, u_4] \) and let \( S \) be the subspace of \( V \) generated by \( \alpha_1 = (1, 1, 2, 3) \), \( \alpha_2 = (2, 3, 5, 7) \) and \( \alpha_3 = (2, 1, 3, 5) \).

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3.3.4 continued

a. If \((x_1, x_2, x_3, x_4) \in S\), how are \(x_3\) and \(x_4\) related to \(x_1\) and \(x_2\)?

b. With \(V\) as above, let \(T\) be the subspace of \(V\) generated by 
   \(a_4 = (1,2,2,3), a_5 = (2,5,4,7),\) and \(a_6 = (3,7,7,8)\).
   If \((x_1, x_2, x_3, x_4) \in T\), how is \(x_4\) expressed in terms of \(x_1, x_2,\) and \(x_3\)?

c. Describe the subspace \(S \cap T\).

d. Describe the subspace \(S + T = \{s + t: s \in S\text{ and } t \in T\}\).

e. Verify that in this example, \(\dim(S + T) = \dim S + \dim T - \dim S \cap T\).

3.3.5

Let \(V = [u_1, u_2, u_3, u_4, u_5]\). Let \(S\) be the subspace spanned by 
\((1,1,2,3,3), (2,3,4,5,7),\) and \((3,4,7,8,8)\), and let \(T\) be the sub-
space spanned by \((1,1,1,3,5), (1,2,3,2,2),\) and \((2,3,3,7,8)\).

a. Find the dimension of \(S\).

b. Find the dimension of \(T\).

c. Find the dimension of \(S \cap T\), and, in particular, find a row reduced 
basis for \(S \cap T\).

d. Find the dimension of \(S + T\) and in particular show how \(x_1, x_2, x_3,\) 
\(x_4,\) and \(x_5\) must be related if \((x_1, x_2, x_3, x_4, x_5) \in S + T\).

e. Again verify that \(\dim(S + T) = \dim S + \dim T - \dim S \cap T\).

3.3.6 (optional)

Our main aim in this exercise is to show how one constructs a 
basis for \(S + T\) by starting with a basis for \(S \cap T\). In the course 
of this construction, we manage to prove that if \(S\) and \(T\) are 
subspaces of a finite dimensional space \(V\), then \(\dim(S + T) = 
\dim S + \dim T - \dim (S \cap T)\).

Use the result of the previous exercise to obtain a basis for 
\(S \cap T\) and then show how this may be augmented by the given basis vectors for \(S\) to form a new basis for \(S\). Apply a similar 
approach to find a new basis for \(T\) and then explain why 

\[\dim (S + T) = \dim S + \dim T - \dim (S \cap T).\]
Resource: Calculus Revisited: Complex Variables, Differential Equations, and Linear Algebra
Prof. Herbert Gross

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