MODE SUPERPOSITION ANALYSIS; TIME HISTORY

LECTURE 11
48 MINUTES
LECTURE 11  Solution of dynamic response by mode superposition

The basic idea of mode superposition

Derivation of decoupled equations

Solution with and without damping

Caughey and Rayleigh damping

Calculation of damping matrix for given damping ratios

Selection of number of modal coordinates

Errors and use of static correction

Practical considerations

TEXTBOOK: Sections: 9.3.1, 9.3.2, 9.3.3

Examples: 9.6, 9.7, 8.9, 9.9, 9.10, 9.11
Mode Superposition Analysis

Basic idea is:

transform dynamic equilibrium
equations into a more effective
form for solution,
using

\[
\mathbf{U} = \mathbf{P} \mathbf{X}(t)
\]

\[
\mathbf{P} = \text{transformation matrix}
\]

\[
\mathbf{X}(t) = \text{generalized displacements}
\]

Using

\[
\underline{U}(t) = \mathbf{P} \mathbf{X}(t) \quad (9.30)
\]

on

\[
\underline{M} \ddot{\mathbf{U}} + \underline{C} \dot{\mathbf{U}} + \underline{K} \mathbf{U} = \underline{R} \quad (9.1)
\]

we obtain

\[
\ddot{\mathbf{X}}(t) + \ddot{\mathbf{C}} \mathbf{X}(t) + \ddot{\mathbf{K}} \mathbf{X}(t) = \ddot{\mathbf{R}}(t) \quad (9.31)
\]

where

\[
\ddot{\mathbf{M}} = \mathbf{P}^T \underline{M} \mathbf{P} \quad ; \quad \ddot{\mathbf{C}} = \mathbf{P}^T \underline{C} \mathbf{P} \quad ;
\]

\[
\ddot{\mathbf{K}} = \mathbf{P}^T \underline{K} \mathbf{P} \quad ; \quad \ddot{\mathbf{R}} = \mathbf{P}^T \underline{R} \quad (9.32)
\]
An effective transformation matrix $P$ is established using the displacement solutions of the free vibration equilibrium equations with damping neglected,

$$M \ddot{U} + K U = 0 \quad (9.34)$$

Using

$$U = \Phi \sin \omega (t - t_0) \quad (9.35)$$

we obtain the generalized eigenproblem,

$$K \Phi = \omega^2 M \Phi \quad (9.36)$$

with the $n$ eigensolutions $(\omega_1^2, \Phi_1)$, $(\omega_2^2, \Phi_2), \ldots, (\omega_n^2, \Phi_n)$, and

$$\Phi_i^T M \Phi_j \begin{cases} = 1 & ; \ i = j \\ = 0 & ; \ i \neq j \end{cases} \quad (9.37)$$

$$0 \leq \omega_1^2 \leq \omega_2^2 \leq \omega_3^2 \ldots \leq \omega_n^2 \quad (9.38)$$
Defining

\[ \Phi = [\phi_1, \phi_2, \ldots, \phi_n] \; ; \; \Omega^2 = \begin{bmatrix} \omega_1^2 \\ \omega_2^2 \\ \vdots \\ \omega_n^2 \end{bmatrix} \]  

we can write

\[ K \Phi = M \Phi \Omega^2 \]  

and have

\[ \Phi^T K \Phi = \Omega^2 \; ; \; \Phi^T M \Phi = I \]  

Now using

\[ U(t) = \Phi X(t) \]  

we obtain equilibrium equations that correspond to the modal generalized displacements

\[ \ddot{X}(t) + \Phi^T \Phi \ddot{X}(t) + \Omega^2 X(t) = \Phi^T R(t) \]  

The initial conditions on \( X(t) \) are obtained using (9.42) and the \( M \) - orthonormality of \( \Phi \); i.e., at time 0 we have

\[ 0_X = \Phi^T M 0_U \; ; \; 0_{\dot{X}} = \Phi^T M 0_{\dot{U}} \]
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Analysis with Damping Neglected

\[ \ddot{x}(t) + \omega_i^2 x(t) = \phi_i^T R(t) \quad (9.45) \]

i.e., \( n \) individual equations of the form

\[ \ddot{x}_i(t) + \omega_i^2 x_i(t) = r_i(t) \quad (i = 1, 2, \ldots, n) \]

where

\[ r_i(t) = \phi_i^T R(t) \quad (9.46) \]

with

\[ x_i \big|_{t=0} = \phi_i^T M ^0 U \quad (9.47) \]

\[ \dot{x}_i \big|_{t=0} = \phi_i^T M ^0 \dot{U} \]

Using the Duhamel integral we have

\[ x_i(t) = \frac{1}{\omega_i} \int_0^t r_i(\tau) \sin \omega_i(t - \tau) d\tau \quad (9.48) \]

\[ + \alpha_i \sin \omega_i t + \beta_i \cos \omega_i t \]

where \( \alpha_i \) and \( \beta_i \) are determined from the initial conditions in (9.47).

And then

\[ U(t) = \sum_{i=1}^{n} \phi_i x_i(t) \quad (9.49) \]
Hence we use

\[ U^P = \sum_{i=1}^{p} \phi_i x_i(t) \]

where

\[ U^P \approx U \]

The error can be measured using

\[ e^P(t) = \frac{\| R(t) - (M \ddot{U}^P(t) + K U^P(t)) \|_2}{\| R(t) \|_2} \]

(9.50)
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Static correction

Assume that we used \( p \) modes to obtain \( \Phi^p \), then let

\[
R = \sum_{i=1}^{n} r_i (M \phi_i)
\]

Hence

\[
r_i = \phi_i^T R
\]

Then

\[
\Delta R = R - \sum_{i=1}^{p} r_i (M \phi_i)
\]

and

\[
K \Delta U = \Delta R
\]

Analysis with Damping Included

Recall, we have

\[
\ddot{X}(t) + \phi^T C \phi \dot{X}(t) + \Omega^2 X(t) = \phi^T R(t)
\]

If the damping is proportional

\[
\phi_i^T C \phi_j = 2\omega_i \xi_i \delta_{ij}
\]

and we have

\[
\ddot{x}_i(t) + 2\omega_i \xi_i \dot{x}_i(t) + \omega_i^2 x_i(t) = r_i(t)
\]

\[
i = 1, \ldots, n
\]
A damping matrix that satisfies the relation in (9.51) is obtained using the Caughey series,

\[ C = M \sum_{k=0}^{p-1} a_k [M^{-1} K]^k \]  \hspace{1cm} (9.56)

where the coefficients \( a_k, k = 1, \ldots, p \), are calculated from the \( p \) simultaneous equations

\[ \xi_i = \frac{1}{2} \left( \frac{a_0}{\omega_i} + a_1 \omega_i + a_2 \omega_i^2 + \ldots \right. \]
\[ + \left. a_{p-1} \omega_i^{p-1} \omega_i^p \right) \]  \hspace{1cm} (9.57)

A special case is Rayleigh damping,

\[ C = \alpha M + \beta K \]  \hspace{1cm} (9.55)

example:

Assume \( \xi_1 = 0.02 \); \( \xi_2 = 0.10 \)
\( \omega_1 = 2 \); \( \omega_2 = 3 \)

calculate \( \alpha \) and \( \beta \)

We use

\[ \phi_1^T (\alpha M + \beta K) \phi_1 = 2\omega_1 \xi_1 \]

or

\[ \alpha + \beta \omega_1^2 = 2\omega_1 \xi_1 \]
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Using this relation for $\omega_1$, $\xi_1$ and $\omega_2$, $\xi_2$, we obtain two equations for $\alpha$ and $\beta$:

\[ \alpha + 4\beta = 0.08 \]
\[ \alpha + 9\beta = 0.60 \]

The solution is $\alpha = -0.336$ and $\beta = 0.104$. Thus the damping matrix to be used is

\[ C = -0.336 \mathbf{M} + 0.104 \mathbf{K} \]

Note that since

\[ \alpha + \beta \omega_i^2 = 2\omega_i \xi_i \]

for any $i$, we have, once $\alpha$ and $\beta$ have been established,

\[ \xi_i = \frac{\alpha + \beta \omega_i^2}{2\omega_i} \]

\[ = \frac{\alpha}{2\omega_i} + \frac{\beta}{2} \omega_i \]
Response solution

As in the case of no damping, we solve $p$ equations

$$\ddot{x}_i + 2\omega_i \xi_i \dot{x}_i + \omega_i^2 x_i = r_i$$

with

$$r_i = \phi_i^T R$$

$$\left. x_i \right|_{t=0} = \phi_i^T M \dot{U}$$

$$\left. \dot{x}_i \right|_{t=0} = \phi_i^T M \ddot{U}$$

and then

$$\mathbf{u}^p = \sum_{i=1}^{p} \phi_i x_i(t)$$

Practical considerations

Mode superposition analysis is effective

- when the response lies in a few modes only, \( p \ll n \)

- when the response is to be obtained over many time intervals (or the modal response can be obtained in closed form).

  e.g. earthquake engineering vibration excitation

- it may be important to calculate \( \varepsilon_p(t) \) or the static correction.