LECTURE 8  Evaluation of isoparametric element matrices

Numerical integrations, Gauss, Newton-Cotes formulas

Basic concepts used and actual numerical operations performed

Practical considerations

Required order of integration, simple examples

Calculation of stresses

Recommended elements and integration orders for one-, two-, three-dimensional analysis, and plate and shell structures

Modeling considerations using the elements

TEXTBOOK: Sections: 5.7.1, 5.7.2, 5.7.3, 5.7.4, 5.8.1, 5.8.2, 5.8.3

Examples: 5.28, 5.29, 5.30, 5.31, 5.32, 5.33, 5.34, 5.35, 5.36, 5.37, 5.38, 5.39
NUMERICAL INTEGRATION,
SOME MODELING CONSIDERATIONS

- Newton-Cotes formulas
- Gauss integration
- Practical considerations
- Choice of elements

We had

\[ K = \int_V \mathbf{B}^T \mathbf{C} \mathbf{B} \, dV \quad (4.29) \]
\[ M = \int_V \mathbf{H}^T \mathbf{H} \, dV \quad (4.30) \]
\[ R_B = \int_V \mathbf{H}^T \mathbf{f}^B \, dV \quad (4.31) \]
\[ R_S = \int_S \mathbf{H}^S \mathbf{f}^S \, dS \quad (4.32) \]
\[ R_I = \int_V \mathbf{B}^T \mathbf{\tau}^I \, dV \quad (4.33) \]
In isoparametric finite element analysis we have:

- the displacement interpolation matrix \( \mathbf{H}(r,s,t) \)
- the strain-displacement interpolation matrix \( \mathbf{B}(r,s,t) \)

Where \( r,s,t \) vary from \(-1\) to \(+1\).

Hence we need to use:

\[
dV = \det \mathbf{J} \, dr \, ds \, dt
\]

Hence, we now have, for example in two-dimensional analysis:

\[
K = \int_{-1}^{+1} \int_{-1}^{+1} \mathbf{B}^T \mathbf{C} \mathbf{B} \det \mathbf{J} \, dr \, ds
\]

\[
M = \int_{-1}^{+1} \int_{-1}^{+1} \rho \mathbf{H}^T \mathbf{H} \det \mathbf{J} \, dr \, ds
\]

etc...
The evaluation of the integrals is carried out effectively using numerical integration, e.g.:

\[ K = \sum_i \sum_j \alpha_{ij} E_{ij} \]

where

- \( i, j \) denote the integration points
- \( \alpha_{ij} \) = weight coefficients
- \( E_{ij} = B_{ij}^T \mathbf{C} B_{ij} \det J_{ij} \)

\[ r = \pm 0.577 \quad s = \pm 0.577 \]

\[ r = \pm 0.775 \quad s = \pm 0.775 \]

\[ r = 0 \quad s = 0 \]

2x2 - point integration
Consider one-dimensional integration and the concept of an interpolating polynomial.
Numerical integrations, modeling considerations

In Newton–Cotes integration we use sampling points at equal distances, and

\[ \int_{a}^{b} F(r) \, dr = (b-a) \sum_{i=0}^{n} C_i^n \, f_i + R_n \]

(5.123)

- \( n \) = number of intervals
- \( C_i^n \) = Newton–Cotes constants
- interpolating polynomial is of order \( n \).
Numerical integrations, modeling considerations

Table 5.1. Newton-Cotes numbers and error estimates.

<table>
<thead>
<tr>
<th>Number of Intervals n</th>
<th>C₀</th>
<th>C₁</th>
<th>C₂</th>
<th>C₃</th>
<th>C₄</th>
<th>C₅</th>
<th>C₆</th>
<th>Upper Bound on Error Rₙ as a Function of the Derivative of F</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1/2</td>
<td>1/2</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>$10^{-3}(b-a)^3 F''(r)$</td>
</tr>
<tr>
<td>2</td>
<td>1/6</td>
<td>4/6</td>
<td>1/6</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>$10^{-3}(b-a)^3 F''(r)$</td>
</tr>
<tr>
<td>3</td>
<td>1/8</td>
<td>3/8</td>
<td>3/8</td>
<td>1/8</td>
<td>1</td>
<td></td>
<td></td>
<td>$10^{-4}(b-a)^4 F''(r)$</td>
</tr>
<tr>
<td>4</td>
<td>7</td>
<td>32</td>
<td>12</td>
<td>32</td>
<td>7</td>
<td></td>
<td></td>
<td>$10^{-3}(b-a)^3 F''(r)$</td>
</tr>
<tr>
<td>5</td>
<td>50</td>
<td>50</td>
<td>50</td>
<td>50</td>
<td>50</td>
<td>50</td>
<td>50</td>
<td>$10^{-4}(b-a)^4 F''(r)$</td>
</tr>
<tr>
<td>6</td>
<td>19</td>
<td>75</td>
<td>50</td>
<td>50</td>
<td>75</td>
<td>19</td>
<td></td>
<td>$10^{-4}(b-a)^4 F''(r)$</td>
</tr>
<tr>
<td></td>
<td>288</td>
<td>288</td>
<td>288</td>
<td>288</td>
<td>288</td>
<td>288</td>
<td>288</td>
<td>$10^{-4}(b-a)^4 F''(r)$</td>
</tr>
<tr>
<td></td>
<td>41</td>
<td>216</td>
<td>27</td>
<td>27</td>
<td>27</td>
<td>27</td>
<td>27</td>
<td>$10^{-4}(b-a)^4 F''(r)$</td>
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<tr>
<td></td>
<td>840</td>
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<td>840</td>
<td>840</td>
<td>840</td>
<td>840</td>
<td>840</td>
<td>$10^{-4}(b-a)^4 F''(r)$</td>
</tr>
</tbody>
</table>

In Gauss numerical integration we use

$$\int_{a}^{b} F(r) \, dr = \alpha_1 F(r_1) + \alpha_2 F(r_2) + \ldots + \alpha_n F(r_n) + R_n \quad (5.124)$$

where both the weights $\alpha_1, \ldots, \alpha_n$ and the sampling points $r_1, \ldots, r_n$ are variables.

The interpolating polynomial is now of order $2n - 1$. 
Numerical integrations, modeling considerations

**Table 5.2.** Sampling points and weights in Gauss-Legendre numerical integration.

<table>
<thead>
<tr>
<th>n</th>
<th>( r_i )</th>
<th>( \alpha_i )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0. (15 zeros)</td>
<td>2. (15 zeros)</td>
</tr>
<tr>
<td>2</td>
<td>( \pm 0.57735 ) 02691 89626</td>
<td>1.00000 00000 00000</td>
</tr>
<tr>
<td>3</td>
<td>( \pm 0.77459 ) 66692 41483</td>
<td>0.55555 55555 55556</td>
</tr>
<tr>
<td>4</td>
<td>( \pm 0.86113 ) 63115 94053</td>
<td>0.34785 48451 34754</td>
</tr>
<tr>
<td>5</td>
<td>( \pm 0.90617 ) 98459 38664</td>
<td>0.23692 68850 56189</td>
</tr>
<tr>
<td>6</td>
<td>( \pm 0.93246 ) 95142 03152</td>
<td>0.17132 44023 79170</td>
</tr>
<tr>
<td></td>
<td>( \pm 0.96120 ) 93864 66265</td>
<td>0.13076 62730 48139</td>
</tr>
<tr>
<td></td>
<td>( \pm 0.23861 ) 91860 83197</td>
<td>0.46791 39345 72691</td>
</tr>
</tbody>
</table>

Now let,

\[ r_i \] be a sampling point and

\[ \alpha_i \] be the corresponding weight

for the interval \(-1\) to \(+1\).

Then the actual sampling point and weight for the interval \(a\) to \(b\) are

\[ \frac{a+b}{2} + \frac{b-a}{2} r_i \] and \( \frac{b-a}{2} \alpha_i \)

and the \( r_i \) and \( \alpha_i \) can be tabulated as in Table 5.2.
Numerical integrations, modeling considerations

In two- and three-dimensional analysis we use

\[
\int_{-1}^{+1} \int_{-1}^{+1} F(r,s) \, dr \, ds = \sum_i \alpha_i \int_{-1}^{+1} F(r_i,s) \, ds
\]

or

\[
\int_{-1}^{+1} \int_{-1}^{+1} F(r,s) \, dr \, ds = \sum_{i,j} \alpha_i \alpha_j F(r_i,s_j)
\]

(5.131)

and corresponding to (5.113),

\[
\alpha_{ij} = \alpha_i \alpha_j, \quad \text{where } \alpha_i \text{ and } \alpha_j
\]

are the integration weights for one-dimensional integration.

Similarly,

\[
\int_{-1}^{+1} \int_{-1}^{+1} \int_{-1}^{+1} F(r,s,t) \, dr \, ds \, dt
\]

\[
= \sum_{i,j,k} \alpha_i \alpha_j \alpha_k F(r_i,s_j,t_k)
\]

(5.133)

and \( \alpha_{ijk} = \alpha_i \alpha_j \alpha_k \).
Numerical integrations, modeling considerations

Practical use of numerical integration

- The integration order required to evaluate a specific element matrix exactly can be evaluated by studying the function $F$ to be integrated.

- In practice, the integration is frequently not performed exactly, but the integration order must be high enough.

Considering the evaluation of the element matrices, we note the following requirements:

a) stiffness matrix evaluation:

   (1) the element matrix does not contain any spurious zero energy modes (i.e., the rank of the element stiffness matrix is not smaller than evaluated exactly); and

   (2) the element contains the required constant strain states.

b) mass matrix evaluation:

   the total element mass must be included.

c) force vector evaluations:

   the total loads must be included.
Demonstrative example

2x2 Gauss integration
"absurd" results

3x3 Gauss integration
correct results

Fig. 5.46. 8 - node plane stress
element supported at B by a
spring.

Stress calculations

\[
\tau = C B U + \tau^I \quad (5.136)
\]

- stresses can be calculated at any point of the element.
- stresses are, in general, discontinuous across element boundaries.
Numerical integrations, modeling considerations

- thickness = 1 cm
- \( E = 3 \times 10^7 \) N/cm²
- \( \nu = 0.3 \)
- \( P = 300 \) N

(a) Cantilever subjected to bending moment and finite element solutions.

Fig. 5.47. Predicted longitudinal stress distributions in analysis of cantilever.
Numerical integrations, modeling considerations

\[ E = 3 \times 10^7 \text{ N/cm}^2 \]
\[ \nu = 0.3 \]
\[ P = 100\text{N} \]

(b) Cantilever subjected to tip-shear force and finite element solutions

Fig. 5.47. Predicted longitudinal stress distributions in analysis of cantilever.

Some modeling considerations

We need

- a qualitative knowledge of the response to be predicted
- a thorough knowledge of the principles of mechanics and the finite element procedures available
- parabolic/undistorted elements usually most effective
Table 5.6 Elements usually effective in analysis.

<table>
<thead>
<tr>
<th>TYPE OF PROBLEM</th>
<th>ELEMENT</th>
</tr>
</thead>
<tbody>
<tr>
<td>TRUSS OR CABLE</td>
<td>2-node</td>
</tr>
<tr>
<td>TWO-DIMENSIONAL</td>
<td></td>
</tr>
<tr>
<td>PLANE STRESS</td>
<td>8-node or</td>
</tr>
<tr>
<td>PLANE STRAIN</td>
<td>9-node</td>
</tr>
<tr>
<td>AXISYMMETRIC</td>
<td></td>
</tr>
<tr>
<td>THREE-DIMENSIONAL</td>
<td>20-node</td>
</tr>
<tr>
<td>3-D BEAM</td>
<td>3-node or</td>
</tr>
<tr>
<td></td>
<td>4-node</td>
</tr>
<tr>
<td>PLATE</td>
<td>9-node</td>
</tr>
<tr>
<td>SHELL</td>
<td>9-node or</td>
</tr>
<tr>
<td></td>
<td>16-node</td>
</tr>
</tbody>
</table>
Numerical integrations, modeling considerations

Fig. 5.49. Some transitions with compatible element layouts
Resource: Finite Element Procedures for Solids and Structures
Klaus-Jürgen Bathe

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