Solution of Nonlinear Dynamic Response—Part II

Contents:

- Mode superposition analysis in nonlinear dynamics
- Substructuring in nonlinear dynamics, a schematic example of a building on a flexible foundation
- Study of analyses to demonstrate characteristics of procedures for nonlinear dynamic solutions
- Example analysis: Wave propagation in a rod
- Example analysis: Dynamic response of a three degree of freedom system using the central difference method
- Example analysis: Ten-story tapered tower subjected to blast loading
- Example analysis: Simple pendulum undergoing large displacements
- Example analysis: Pipe whip solution
- Example analysis: Control rod drive housing with lower support
- Example analysis: Spherical cap under uniform pressure loading
- Example analysis: Solution of fluid-structure interaction problem

Textbook: Sections 9.3.1, 9.3.2, 9.3.3, 9.5.3, 8.2.4
Examples: 9.6, 9.7, 9.8, 9.11
The use of the nonlinear dynamic analysis techniques is described with example solutions in


THE SOLUTION OF
THE DYNAMIC EQUILIBRIUM EQUATIONS CAN
BE ACHIEVED USING
- DIRECT INTEGRATION
  - EXPLICIT INTEGRATION
  - IMPLICIT INTEGRATION
- MODE SUPERPOSITION
- SUBSTRUCTURING

WE DISCUSS THESE TECHNIQUES BRIEFLY IN THIS LECTURE

EXAMPLES

| EX.1 | WAVE PROPAGATION IN A ROD |
| EX.2 | RESPONSE OF A 3 DOF SYSTEM |
| EX.3 | ANALYSIS OF TEN STORY TAPERED TOWER |
| EX.4 | ANALYSIS OF PENDULUM |
| EX.5 | PIPE WHIP RESPONSE SOLUTION |

SLIDES REGARDING

- ANALYSIS OF CRD HOUSING
- SOLUTION OF RESPONSE OF SPHERICAL CAP
- ANALYSIS OF FLUID-STRUCTURE INTERACTION PROBLEM (PIPE TEST)

THE DETAILS OF THESE PROBLEM SOLUTIONS ARE GIVEN IN THE PAPERS, SEE STUDY GUIDE
Mode superposition:

- The modes of vibration change due to the nonlinearities, however we can employ the modes at a particular time as basis vectors (generalized displacements) to express the response.

- This method is effective when, in nonlinear analysis,
  - the response lies in only a few vibration modes (displacement patterns)
  - the system has only local nonlinearities

The governing equations in implicit time integration are (assuming no damping matrix)

\[ M(t+\Delta t)\dot{U}^{(k)} + \tau K\Delta U^{(k)} = t+\Delta tR - t+\Delta tF^{(k-1)} \]

Let now \( \tau = 0 \), hence the method of solution corresponds to the initial stress method.

Using

\[ t+\Delta tU = \sum_{i=r}^{s} \phi_i \Delta x_i \]

\[ 0K \phi_i = \omega_i^2 M \phi_i \]
The modal transformation gives
\[ t + \Delta t \ddot{X}^{(k)} + \Omega^2 \Delta X^{(k)} = \Phi^T \left( t + \Delta t R - t + \Delta t F^{(k-1)} \right) \]
where
\[ \Omega^2 = \begin{bmatrix} \omega_r^2 & \omega_s^2 \end{bmatrix} \]
\[ \Phi = [\phi_r \ldots \phi_s] \]
\[ t + \Delta t X^T = [t + \Delta t x_r \ldots t + \Delta t x_s] \]

Typical problem:

Pipe whip: Elastic-plastic pipe
Elastic-plastic stop

- Nonlinearities in pipe and stop. But the displacements are reasonably well contained in a few modes of the linear (initial) system.
Substructuring

- Procedure is used with implicit time integration. All linear degrees of freedom can be condensed out prior to the incremental solution.
- Used for local nonlinearities:
  - Contact problems
  - Nonlinear support problems

Example:

- "master" node
- substructure internal node

Ten story building

Finite element model

Substructure model
Here

\[ t\hat{K} = \left( K + \frac{4}{\Delta t^2} M \right) + t'K_{\text{nonlinear}} \]

\[ = \hat{K} + t'K_{\text{nonlinear}} \]
After condensing out all substructure internal degrees of freedom, we obtain a smaller system of equations:

$$\begin{align*}
\mathbf{t}_c \mathbf{K}_c &= \mathbf{K}_c + \mathbf{t}_c \mathbf{K}_{\text{nonlinear}} \\
\Delta \mathbf{U}_c^{(n)} &= \Delta \mathbf{U}_c^{(0)}
\end{align*}$$

Major steps in solution:

- Prior to step-by-step solution, establish $\mathbf{K}$ for all mass and constant stiffness contributions. Statically condense out internal substructure degrees of freedom to obtain $\mathbf{K}_c$.

We note that

$$\mathbf{t}_c \mathbf{K}_c = \mathbf{K}_c + \mathbf{t}_c \mathbf{K}_{\text{nonlinear}}$$

from $\mathbf{K} = \mathbf{K} + \frac{4}{\Delta t^2} \mathbf{M}$

all linear element contributions  total mass matrix
For each time step solution (and each equilibrium iteration):

- Update condensed matrix, \( \tilde{K}_c \), for nonlinearities.
- Establish complete load vector for all degrees of freedom and condense out substructure internal degrees of freedom.
- Solve for master dof displacements, velocities, accelerations and calculate all substructure dof disp., vel., acc.

The substructure internal nodal disp., vel., acc. are needed to calculate the complete load vector (corresponding to all dof).

Solution procedure for each time step (and iteration):
Example: Wave propagation in a rod

Uniform, freely floating rod

\( R \)

\( \text{L} = 1.0 \text{ m} \)
\( A = 0.01 \text{ m}^2 \)
\( \rho = 1000 \text{ kg/m}^3 \)
\( E = 2.0 \times 10^9 \text{ Pa} \)

Consider the compressive force at a point at the center of the rod:

\( \text{Compressive force} \)

1000 N

\( \frac{1}{2} t^* \quad t^* \quad \frac{3}{2} t^* \quad 2t^* \)

\( t^* = \text{time for stress wave to travel through the rod} \)
We now use a finite element mesh of ten 2-node truss elements to obtain the compressive force at point A.

All elements uniformly spaced

Central difference method:

- The critical time step for this problem is
  \[ \Delta t_{cr} = \frac{L_e}{c} = L e \left( \frac{1}{\text{number of elements}} \right) \]
  \[ \Delta t > \Delta t_{cr} \] will produce an unstable solution

- We need to use the initial conditions as follows:
  \[ \dot{M} \dot{U} + K \ddot{U} = 0 \]
  \[ \dot{U}_i = \frac{0 \dot{R}_i}{m_i} \]
• Using a time step equal to $\Delta t_{cr}$, we obtain the correct result:

- For this special case the exact solution is obtained.

Finite elements

Compressive force (N)

$-500$ $0$ $500$ $1000$ $1500$

$2t^*$ time

t*$

Transparency 14-17

• Using a time step equal to $\frac{1}{2} \Delta t_{cr}$, the solution is stable, but highly inaccurate.

Finite elements

Compressive force (N)

$-500$ $0$ $500$ $1000$ $1500$

$2t^*$ time

t*$

Transparency 14-18
Now consider the use of the trapezoidal rule:

- A stable solution is obtained with any choice of $\Delta t$.
- Either a consistent or lumped mass matrix may be used. We employ a lumped mass matrix in this analysis.

Trapezoidal rule, $\Delta t = \Delta t_{cr}^{cdm}$, initial conditions computed using $M^0 \ddot{U} = 0 R$.

- The solution is inaccurate.
Trapezoidal rule, $\Delta t = \Delta t_{cr|CDM}$, zero initial conditions.

— Almost same solution is obtained.

Finite element solution, 10 element mesh

exact solution

Compressive force (N)

Trapezoidal rule, $\Delta t = 2\Delta t_{cr|CDM}$

— The solution is stable, although inaccurate.

Finite element solution, 10 element mesh

exact solution
Trapezoidal rule, $\Delta t = \frac{1}{2} \Delta t_{cr\mid_{CDM}}$

The same phenomena are observed when a mesh of one hundred 2-node truss elements is employed.

Here $\Delta t_{cr} = t^*/100$
Trapezoidal rule, $\Delta t = \Delta t_{cr\left|_{CDM}}$

Finite element solution, 100 element mesh

Now consider a two-dimensional model of the rod:

- thickness = 0.2 m
- $E = 2 \times 10^9$ Pa
- $\nu = 0$
- $\rho = 1000$ kg/m$^3$

For this mesh, $\Delta t_{cr} \neq t^*/(10$ elements) because the element width is less than the element length.
If $\Delta t = t^* / (10 \text{ elements})$ is used, the solution diverges

- In element 5,
  \[ |\tau_{zz}| > \left( \frac{1000 \text{ N}}{0.01 \text{ m}^2} \right) \]
  at $t = 1.9 t^*$

Example: Dynamic response of three degree-of-freedom system using central difference method

\[ k_L = 1 \text{ lbf/ft} \]
\[ m = 1 \text{ slug} \]
\[ \omega x_1 = \omega x_2 = \omega x_3 = 0 \]
\[ \omega x_1 = 0.555 \text{ ft/sec} \]
\[ \omega x_2 = 1.000 \text{ ft/sec} \]
\[ \omega x_3 = 1.247 \text{ ft/sec} \]

(\( \Delta t_{\text{crit}} \))linear = 1.11 sec
(\( \Delta t_{\text{crit}} \)nonlinear = 0.14 sec
Results: Response of right mass

Response of center mass:
Response of left mass:

\[
\begin{array}{ccc}
\text{Disp. (ft)} & X_3 & t(\text{sec}) \\
10 & 15 & 20 \\
0 & 0 & 0 \\
-1 & -1 & -1 \\
\end{array}
\]

\[\bullet: \Delta = 0.05 \text{ sec.} \]
\[\circ: \Delta = 0.15 \text{ sec.}\]

Force (lbf) in center truss:

<table>
<thead>
<tr>
<th>TIME</th>
<th>(\Delta t = 0.05)</th>
<th>(\Delta t = 0.15)</th>
</tr>
</thead>
<tbody>
<tr>
<td>9.0</td>
<td>-0.666</td>
<td>-0.700</td>
</tr>
<tr>
<td>12.0</td>
<td>-0.804</td>
<td>-0.877</td>
</tr>
<tr>
<td>15.0</td>
<td>0.504</td>
<td>0.503</td>
</tr>
<tr>
<td>18.0</td>
<td>0.648</td>
<td>-0.100</td>
</tr>
<tr>
<td>21.0</td>
<td>-0.132</td>
<td>-0.059</td>
</tr>
<tr>
<td>24.0</td>
<td>-0.922</td>
<td>0.550</td>
</tr>
</tbody>
</table>
Example: 10 story tapered tower

Girder properties:
- \( E = 2.07 \times 10^{11} \) Pa
- \( v = 0.3 \)
- \( A = 0.01 \) m
- \( A_s = 0.009 \) m
- \( l = 8.33 \times 10^{-5} \) m
- \( \rho = 7800 \) kg/m

Force per unit length (N/m):

![Graph showing force per unit length over time (milliseconds)](image_url)
Purpose of analysis:

- Determine displacements, velocities at top of tower.
- Determine moments at base of tower.

We use the trapezoidal rule and a lumped mass matrix in the following analysis.

We must make two decisions:

- Choose mesh (specifically the number of elements employed).
- Choose time step $\Delta t$.

These two choices are closely related:

The mesh and time step to be used depend on the loading applied.
Some observations:

- The choice of mesh determines the highest natural frequency (and corresponding mode shape) that is accurately represented in the finite element analysis.

- The choice of time step determines the highest frequency of the finite element mesh in which the response is accurately integrated during the time integration.

- Hence, it is most effective to choose the mesh and time step such that the highest frequency accurately "integrated" is equal to the highest frequency accurately represented by the mesh.

- The applied loading can be represented as a Fourier series which displays the important frequencies to be accurately represented by the mesh.
Consider the Fourier representation of the load function:

\[ f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos(2\pi f_n t) + b_n \sin(2\pi f_n t)) \]

Including terms up to

- case 1: \( f_n = 17 \text{ Hz} \)
- case 2: \( f_n = 30 \text{ Hz} \)

The loading function is represented as shown next.

Fourier approximation including terms up to 17 Hz:
Fourier approximation including terms up to 30 Hz:

- We choose a 30 element mesh, a 60 element mesh and a 120 element mesh. All elements are 2-node Hermitian beam elements.
Determine "accurate" natural frequencies represented by 30 element mesh:

From eigenvalue solutions of the 30 and 60 element meshes, we find

<table>
<thead>
<tr>
<th>mode number</th>
<th>natural frequencies (Hz)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>30 element mesh</td>
</tr>
<tr>
<td>1</td>
<td>1.914</td>
</tr>
<tr>
<td>2</td>
<td>4.815</td>
</tr>
<tr>
<td>3</td>
<td>8.416</td>
</tr>
<tr>
<td>4</td>
<td>12.38</td>
</tr>
<tr>
<td>5</td>
<td>16.79</td>
</tr>
<tr>
<td>6</td>
<td>21.45</td>
</tr>
<tr>
<td>7</td>
<td>26.18</td>
</tr>
<tr>
<td>8</td>
<td>30.56</td>
</tr>
</tbody>
</table>

Calculate time step:

\[
T_{co} = \frac{1}{17} \text{Hz} = .059 \text{ sec}
\]

\[
\Delta t = \frac{1}{20} T_{co} = .003 \text{ sec}
\]

- A smaller time step would accurately "integrate" frequencies, which are not accurately represented by the mesh.

- A larger time step would not accurately "integrate" all frequencies which are accurately represented by the mesh.
Determine "accurate" natural frequencies represented by 60 element mesh:

From eigenvalue solutions of the 60 and 120 element meshes, we find

<table>
<thead>
<tr>
<th>mode number</th>
<th>natural frequencies (Hz)</th>
<th>60 element mesh</th>
<th>120 element mesh</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>17.27</td>
<td>17.28</td>
<td></td>
</tr>
<tr>
<td>6</td>
<td>22.47</td>
<td></td>
<td>22.49</td>
</tr>
<tr>
<td>7</td>
<td>28.08</td>
<td></td>
<td>28.14</td>
</tr>
<tr>
<td>8</td>
<td>29.80</td>
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<td>29.75</td>
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<td>9</td>
<td>32.73</td>
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<td>33.85</td>
</tr>
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<td>10</td>
<td>33.73</td>
<td></td>
<td>35.06</td>
</tr>
<tr>
<td>11</td>
<td>36.30</td>
<td></td>
<td>38.96</td>
</tr>
</tbody>
</table>

Calculate time step:

$$T_{co} = \frac{1}{30} \text{Hz} = .033 \text{ sec}$$

$$\Delta t \doteq \frac{1}{20} T_{co} = .0017 \text{ sec}$$

- The meshes chosen correspond to the Fourier approximations discussed earlier:

  30 element mesh \[\longrightarrow\] Fourier approximation including terms up to 17 Hz.

  60 element mesh \[\longrightarrow\] Fourier approximation including terms up to 30 Hz.
Pictorially, at time 200 milliseconds, we have (note that the displacements are amplified for visibility):

<table>
<thead>
<tr>
<th>30 elements</th>
<th>60 elements</th>
</tr>
</thead>
<tbody>
<tr>
<td><img src="image1" alt="Diagram" /></td>
<td><img src="image2" alt="Diagram" /></td>
</tr>
</tbody>
</table>

Pictorially, at time 400 milliseconds, we have (note that the displacements are amplified for visibility):

<table>
<thead>
<tr>
<th>30 elements</th>
<th>60 elements</th>
</tr>
</thead>
<tbody>
<tr>
<td><img src="image3" alt="Diagram" /></td>
<td><img src="image4" alt="Diagram" /></td>
</tr>
</tbody>
</table>
Consider the moment reaction at the base of the tower:

\[
M \quad \text{(KN·m)}
\]

\[
\begin{array}{c}
\text{time (milliseconds)} \\
0 & 250 & 500
\end{array}
\]

- : 30 elements
- : 60 elements

Consider the horizontal displacement at the top of the tower:

\[
\begin{array}{c}
u \\
\text{time (milliseconds)} \\
0 & 250 & 500
\end{array}
\]

- : 30 elements
- : 60 elements
Consider the horizontal velocity at the top of the tower:

\[ V \text{ (m/s)} \]

-2 -1 0 1 2 3

\[ \text{time (milliseconds)} \]

60 elements

30 elements

Comments:

- The high-frequency oscillation observed in the moment reaction from the 60 element mesh is probably inaccurate. We note that the frequency of the oscillation is about 110 Hz (this can be seen directly from the graph).

- The obtained solutions for the horizontal displacement at the top of the tower are virtually identical.
Example: Simple pendulum undergoing large displacements

\[ g = 980 \text{ cm/sec}^2 \]

- Initial conditions:
  \[ \theta_0 = 90^\circ \]
  \[ \dot{\theta}_0 = 0 \]

One truss element with tip concentrated mass is employed.

Calculation of dynamic response:
- The trapezoidal rule is used to integrate the time response.
- Full Newton iterations are used to reestablish equilibrium during every time step.
- Convergence tolerance:
  \[ \text{ETOL} = 10^{-7} \]
  (a tight tolerance)
Choose \( \Delta t = 0.1 \text{ sec} \). The following response is obtained:

![Graph showing \( \theta \) vs. time with an expected solution and last obtained solution marked.]

The strain in the truss is plotted:

- An instability is observed.

![Graph showing strain vs. time with values of \( 10 \times 10^{-5} \), \( 5 \times 10^{-5} \), and \( -5 \times 10^{-5} \).]
• The instability is unchanged when we tighten our convergence tolerances.
• The instability is also observed when the BFGS algorithm is employed.
• Recall that the trapezoidal rule is unconditionally stable only in linear analysis.

Choose $\Delta t = 0.025$ sec, using the original tolerance and the full Newton algorithm (without line searches).
• The analysis runs to completion.
The strain in the truss is stable:

finite element solution, $\Delta t = 0.025$ sec

It is important that equilibrium be accurately satisfied at the end of each time step:

Finite element solution, $\Delta t = 0.025$ sec., equilibrium iterations used as described above.

Finite element solution, $\Delta t = 0.025$ sec., no equilibrium iterations used.
Although the solution obtained without equilibrium iterations is highly inaccurate, the solution is stable:

Finite element solution, \( \Delta t=0.025 \) sec., no equilibrium iterations used.

Finite element solution, \( \Delta t=0.025 \) sec., equilibrium iterations used as described above.

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Example: Pipe whip analysis:

- Determine the transient response when a step load \( P \) is suddenly applied.

All dimensions in inches.
Finite element model:

Six Hermitian beam elements

- The truss element incorporates a 3 inch gap.

Material properties:

Pipe: $E = 2.698 \times 10^7$ psi
$\nu = 0.3$
$\sigma_y = 2.914 \times 10^4$ psi
$E_T = 0$
$\rho = 8.62 \times 10^{-3} \text{ slug/in}^3 = 7.18 \times 10^{-4} \text{ lbf-sec}^2/\text{in}^4$

Restraint: $E = 2.99 \times 10^7$ psi
$\sigma_y = 3.80 \times 10^4$ psi
$E_T = 0$
The analysis is performed using
- Mode superposition (2 modes)
- Direct time integration

We use, for each analysis,
- Trapezoidal rule
- Consistent mass matrix

A convergence tolerance of \( ETOL = 10^{-7} \) is employed.

Eigenvalue solution:

Mode 1, natural frequency = 8.5 Hz

Mode 2, natural frequency = 53 Hz
Choice of time step:

We want to accurately integrate the first two modes:

\[ \Delta t = \frac{1}{20} T_{co} = \frac{1}{20} \left( \frac{1}{\text{frequency of mode 2}} \right) \]

= 0.001 sec

Note: This estimate is based solely on a linear analysis (i.e., before the pipe hits the restraint and while the pipe is still elastic).

Determine the tip displacement:

[Graph showing time (milliseconds) vs. tip displacement (in)]

- mode superposition
- direct integration
Determine the moment at the built-in end of the beam:

<table>
<thead>
<tr>
<th>Time (milliseconds)</th>
<th>Moment (lb-in)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>2</td>
<td>-1 x 10^7</td>
</tr>
<tr>
<td>4</td>
<td>-2 x 10^7</td>
</tr>
<tr>
<td>6</td>
<td>-3 x 10^7</td>
</tr>
<tr>
<td>8</td>
<td>-4 x 10^7</td>
</tr>
<tr>
<td>10</td>
<td></td>
</tr>
</tbody>
</table>

- Mode superposition
- Direct integration
Analysis of CRD housing with lower support

CRD housing tip deflection

- PETERSON AND BATHE
- DIRECT INTEGRATION
- MODE SUPERPOSITION (12 MODES)
Ten 8-node axisymmetric els.
Newmark inte (δ = 0.55, α = 0.276)
2 x 2 Gauss integration
consistent mass
Δt = 10μsec, T.L.

Spherical cap nodes under uniform pressure loading

Dynamic elastic-plastic response of a spherical cap,
dereformation independent
Response of the cap using consistent and lumped mass idealization

Effect of numbers of Gauss integration points on the cap response predicted
Slide 14-7

Analysis of fluid—structure interaction problem (pipe test)

Slide 14-8

Pressure pulse input
Slide 14-9

finite element model

Slide 14-10
Resource: Finite Element Procedures for Solids and Structures
Klaus-Jürgen Bathe

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