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- Basic considerations in modeling inelastic response
- A schematic review of laboratory test results, effects of stress level, temperature, strain rate
- One-dimensional stress-strain laws for elasto-plasticity, creep, and viscoplasticity
- Isotropic and kinematic hardening in plasticity
- General equations of multiaxial plasticity based on a yield condition, flow rule, and hardening rule
- Example of von Mises yield condition and isotropic hardening, evaluation of stress-strain law for general analysis
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The plane strain punch problem is also considered in

- We discussed in the previous lectures the modeling of elastic materials
  - Linear stress-strain law
  - Nonlinear stress-strain law
  - Linearly elastic
  - The T.L. and U.L. formulations

- We now want to discuss the modeling of inelastic materials
  - Elasto-plasticity
  - Creep
- We proceed as follows:
  - We discuss briefly inelastic material behaviors, as observed in laboratory tests
  - We discuss briefly modeling of such response in 1-D analysis

- We generalize our modeling considerations to 2-D and 3-D stress situations
MODELING OF INELASTIC RESPONSE: ELASTO-PLASTICITY, CREEP AND VISCOPLASTICITY

- The total stress is not uniquely related to the current total strain. Hence, to calculate the response history, stress increments must be evaluated for each time (load) step and added to the previous total stress.

\[ d\sigma_{ij} = C_{ijrs}^E (de_{rs} - de_{rs}^{in}) \]

where

- \( C_{ijrs}^E \) = components of the elasticity tensor
- \( de_{rs} \) = total differential strain increment
- \( de_{rs}^{in} \) = inelastic differential strain increment
The inelastic response may occur rapidly or slowly in time, depending on the problem of nature considered.

Modeling:
- In plasticity, the model assumes that \( \varepsilon^{\text{pl}} \) occurs instantaneously with the load application.
- In creep, the model assumes that \( \varepsilon^{\text{creep}} \) occurs as a function of time.
- The actual response in nature can be modeled using plasticity and creep together, or alternatively using a viscoplastic material model.

In the following discussion we assume small strain conditions, hence
- we have either a materially-nonlinear-only analysis
- or a large displacement/large rotation but small strain analysis
• As pointed out earlier, for the large displacement solution we would use the total Lagrangian formulation and in the evaluation of the stress-strain laws simply use

- Green-Lagrange strain component for the engineering strain components

and

- 2nd Piola-Kirchhoff stress components for the engineering stress components

Consider a brief summary of some observations regarding material response measured in the laboratory

• We only consider schematically what approximate response is observed; no details are given.

• Note that, regarding the notation, no time, t, superscript is used on the stress and strain variables describing the material behavior.
MATERIAL BEHAVIOR, "INSTANTANEOUS" RESPONSE

Tensile Test: Assume

- small strain conditions
- behavior in compression is the same as in tension

Hence

\[ e = \frac{l - l_0}{l_0} \]

\[ \sigma = \frac{P}{A_0} \]

Assumed engineering stress, \( \sigma \)

Engineering strain, \( e \)

Fracture

Ultimate strain

Test

Constant temperature
Effect of strain rate:

\[ \frac{de}{dt} \text{ increasing} \]

Effect of temperature

\[ \text{temperature is increasing} \]
MATERIAL BEHAVIOR, TIME-DEPENDENT RESPONSE

- Now, at constant stress, inelastic strains develop.

- Important effect for materials when temperatures are high
Effect of stress level on creep strain

$e$ vs. time

$\sigma$ increasing

fracture

$\text{temperature} = \text{constant}$

Effect of temperature on creep strain

$e$ vs. time

fracture

$\sigma = \text{constant}$

temperature increasing
MODELING OF RESPONSE

Consider a one-dimensional situation:

\[ t_u \quad t_\sigma = \frac{P}{A} \]
\[ t_e = \frac{u}{L} \]

- We assume that the load is increased monotonically to its final value, \( P^* \).
- We assume that the time is "long" so that inertia effects are negligible (static analysis).

Load plasticity effects creep effects
\[ P^* \]

predominate

\[ t^* \text{ (small)} \]

time interval without time-dependent inelastic strains

time-dependent inelastic strains are accumulated – modeled as creep strains
Plasticity, uniaxial, bilinear material model:

\[ t^e^C = t^e^E \]
\[ t^e^I^N = t^e^P \]

Creep, power law material model:

\[ e^C = a_0 \sigma^{a1} t^{a2} \]
\[ t^\sigma = E t^e^E \]
\[ t^e^I^N = t^e^C + t^e^P \]

- The elastic strain is the same as in the plastic analysis (this follows from equilibrium).
- The inelastic strain is time-dependent and time is now an actual variable.
Viscoplasticity:

- Time-dependent response is modeled using a fluidity parameter $\gamma$:

$$\dot{e} = \frac{\dot{\sigma}}{E} + \gamma \left( \frac{\sigma}{\sigma_y} - 1 \right) \frac{\sigma}{e^{\varepsilon_{VP}}}$$

where

$$\langle \sigma - \sigma_y \rangle = \begin{cases} 
0 & , \sigma \leq \sigma_y \\
\sigma - \sigma_y & , \sigma > \sigma_y 
\end{cases}$$

Typical solutions (1-D specimen):

Non-hardening material: 
- Total strain increasing $\gamma$
- Elastic strain
- Time

Hardening material: 
- Total strain decreases as function of $\varepsilon_{VP}$
- Elastic strain
- Time
PLASTICITY

• So far we considered only loading conditions.

• Before we discuss more general multiaxial plasticity relations, consider unloading and cyclic loading assuming uniaxial stress conditions.

• Consider that the load increases in tension, causes plastic deformation, reverses elastically, and again causes plastic deformation in compression.
Bilinear material assumption, isotropic hardening

\[ \sigma \quad E \quad \sigma_y \quad \sigma_y^I \]

\[ e \quad e_1^p \quad e_2^p \]

Bilinear material assumption, kinematic hardening

\[ \sigma \quad E \quad 2\sigma_y \]

\[ e \quad e_1^p \quad e_2^p \]
MULTIAXIAL PLASTICITY

To describe the plastic behavior in multiaxial stress conditions, we use

- A yield condition
- A flow rule
- A hardening rule

In the following, we consider isothermal (constant temperature) conditions.

These conditions are expressed using a stress function $t^F$.

Two widely used stress functions are the

von Mises function

Drucker-Prager function
von Mises

\[ 'F = \frac{1}{2} 's_{ij} 's_{ij} - 'k \]

\[ 's_{ij} = '\sigma_{ij} - \frac{1}{2} '\sigma_{mm} \delta_{ij} ; \quad 'k = \frac{1}{3} '\sigma_y^2 \]

Drucker-Prager

\[ 'F = 3\alpha '\sigma_m + '\bar{\sigma} - k \]

\[ '\sigma_m = \frac{1}{3} '\sigma_{ii} ; \quad '\bar{\sigma} = \sqrt{\frac{1}{2} 's_{ij} 's_{ij}} \]

We use both matrix notation and index notation:

\[ \begin{bmatrix} d\varepsilon_{11}^P \\ d\varepsilon_{22}^P \\ d\varepsilon_{33}^P \\ d\varepsilon_{12}^P + d\varepsilon_{21}^P \\ d\varepsilon_{23}^P + d\varepsilon_{32}^P \\ d\varepsilon_{13}^P + d\varepsilon_{31}^P \end{bmatrix}, \quad \begin{bmatrix} d\sigma_{11} \\ d\sigma_{22} \\ d\sigma_{33} \\ d\sigma_{12} \\ d\sigma_{23} \\ d\sigma_{31} \end{bmatrix} \]

matrix notation

note that both \( d\varepsilon_{12}^P \) and \( d\varepsilon_{21}^P \) are added
The basic equations are then (von Mises $^1F$):

1) Yield condition

\[ ^1F (\sigma_{ij}, \kappa) = 0 \]

- $\sigma_{ij}$: current stresses
- $\kappa$: function of plastic strains

$^1F$ is zero throughout the plastic response

- 1-D equivalent: $\frac{1}{3} (\sigma^2 - \sigma_\gamma^2) = 0$
  - (uniaxial stress)
  - $\sigma^2$: current stresses
  - $\sigma_\gamma^2$: function of plastic strains.
2) Flow rule (associated rule):

\[ \text{de}^p_{ij} = t\lambda \frac{\partial F}{\partial \sigma_{ij}} \]

where \( t\lambda \) is a positive scalar.

- 1-D equivalent:

\[ \begin{align*}
\text{de}^p_{11} &= \frac{2}{3} t\lambda \ t \sigma \\
\text{de}^p_{22} &= -\frac{1}{3} t\lambda \ t \sigma \\
\text{de}^p_{33} &= -\frac{1}{3} t\lambda \ t \sigma
\end{align*} \]

3) Stress-strain relationship:

\[ d\sigma = C^E (\text{de} - \text{de}^p) \]

- 1-D equivalent:

\[ d\sigma = E (\text{de}_{11} - \text{de}^p_{11}) \]
Our goal is to determine $C^{EP}$ such that

$$d\sigma = \frac{C^{EP}}{\epsilon^{EP}} \, de$$

instantaneous elastic-plastic stress-strain matrix

General derivation of $C^{EP}$:

Define

$$\dot{q}_{ij} = \frac{\partial \dot{F}}{\partial \sigma_{ij}} \epsilon_{ij}^{EP} \text{ fixed}$$

$$\dot{p}_{ij} = -\frac{\partial \dot{F}}{\partial \epsilon_{ij}} \epsilon_{ij}^{EP} \text{ fixed}$$
Using matrix notation, results from our definition of the plastic strain and stress increment vectors:

\[ \mathbf{q}^T = \begin{bmatrix} q_{11} & q_{22} & q_{33} \\ \hat{q}_{12} & \hat{q}_{23} & \hat{q}_{31} \end{bmatrix} \]

\[ \mathbf{p}^T = \begin{bmatrix} p_{11} & p_{22} & p_{33} \\ \hat{p}_{12} & \hat{p}_{23} & \hat{p}_{31} \end{bmatrix} \]

We now determine \( \mathbf{q}^T \lambda \) in terms of \( \dd e \):

Using \( \dd^T \mathbf{a} = 0 \) during plastic deformations,

\[ d^T \mathbf{F} = \frac{\partial F}{\partial \sigma_{ij}} d\sigma_{ij} + \frac{\partial F}{\partial e_{ij}} d\varepsilon_{ij} \]

\[ = \mathbf{q}^T d\sigma - \mathbf{p}^T (d\varepsilon - \lambda \dd e) \]

\[ = 0 \]
Also

\[ t^q T \, d\sigma = t^q T \left( C^E \left( de - de^P \right) \right) \]

The flow rule assumption may be written as

\[ de^P = \lambda \, t^q \]

Hence

\[ t^q T \, d\sigma = \left[ t^q T \left( C^E \left( de - \lambda \, t^q \right) \right) \right] = \lambda \, t^p T \, t^q \]

from \( d^F = 0 \)

Solving the boxed equation for \( \lambda \) gives

\[ \lambda = \frac{t^q T \, C^E \, de}{p^T \, t^q + t^q \, C^E \, t^q} \]

Hence we can determine the plastic strain increment from the total strain increment:

\[ de^P = \left( \frac{t^q T \, C^E \, de}{p^T \, t^q + t^q \, C^E \, t^q} \right) \, t^q \]
We can now solve for $C^{EP}$:

$$d\sigma = C^E (d\varepsilon - d\varepsilon^P)$$

function of $d\varepsilon$ from above

$$C^{EP} = C^E - \frac{C^E t^q (C^E t^q)^T}{p^T t^q + t^q C^E t^q}$$

Example: Von Mises yield condition, isotropic hardening

Two equivalent equations:

$$'\sigma_y = \frac{\sqrt{2}}{2} \sqrt{(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2}$$

principal stresses

$$'F = \frac{1}{2} 's_{ij} 's_{ij} - 'k; 'k = \frac{1}{3} '\sigma_y^2$$

deviatoric stresses: $'s_{ij} = '\sigma_{ij} - \frac{1}{3} '\sigma_{mm} \delta_{ij}$
We now compute the derivatives of the yield function.

First consider $t\psi$:

$$
t_{\psi} = -\frac{\partial F}{\partial \mathbf{e}_p^{\psi} \mid \sigma_0 \text{ fixed}} = -\frac{\partial \mathbf{e}_p^{\psi}}{\partial \sigma_0} \left( \frac{1}{2} t_s^{\psi} t_s^{\psi} - \frac{1}{3} t_{\sigma_y}^2 \right)
$$

$$
= \frac{2}{3} t_{\sigma_y} \frac{\partial t_{\sigma_y}}{\partial \mathbf{e}_p^{\psi}} \quad (\sigma_0 \text{ fixed implies } t_s^{\psi} \text{ is fixed})
$$
What is the relationship between $\sigma_y$ and the plastic strains?

We answer this question using the concept of "plastic work".

- The plastic work (per unit volume) is the amount of energy that is unrecoverable when the material is unloaded.
- This energy has been used in creating the plastic deformations within the material.

- Pictorially: 1-D example

  \begin{align*}
  \text{stress} & \quad \text{time } t \\
  \text{slope } E & \quad \text{slope } E_T \\
  \text{Shaded area equals plastic work } W_P: & \quad W_P = \int_0^{t_p} \tau \sigma \, \mathrm{d}e^P \\
  \text{In general, } W_P &= \int_0^{t_p} \tau \sigma \, \mathrm{d}e^P
  \end{align*}
Consider 1-D test results: the current yield stress may be written in terms of the plastic work.

\[ W_p = \frac{1}{2} \left( \frac{1}{E_T} - \frac{1}{E} \right) (\gamma^2 - \sigma_y^2) \]

\[ \frac{d\sigma_y}{d\gamma} = \left( \frac{E}{E - E_T} \right) \frac{1}{\sigma_y} \]

We can now evaluate \( t_{p_{ij}} \) — which corresponds to a generalization of the 1-D test results to multiaxial conditions.

\[ t_{p_{ij}} = \frac{2}{3} t_{\sigma_y} \left( \frac{d\sigma_y}{d\gamma} \frac{\partial W_p}{\partial \varepsilon_{ij}^p} \right) \frac{\partial \sigma_y}{\partial \varepsilon_{ij}^p} \]

\[ = \frac{2}{3} t_{\sigma_y} \left( \left( \frac{E}{E - E_T} \right) \frac{1}{\sigma_y} \right) (t_{\sigma_y}) \]

\[ = \frac{2}{3} \left( \frac{E}{E - E_T} \right) t_{\sigma_y} \]
Alternatively, we could have used that
\[ d^t W_p = t\bar{\sigma} \, d^t \bar{\varepsilon}^p \]
where
\[ t\bar{\sigma} = \sqrt{\frac{3}{2} t s_{ij} t s_{ij}} \] (effective stress)
\[ d^t \bar{\varepsilon}^p = \sqrt{\frac{2}{3} d e_{ij} \, d e_{ij}} \] (increment in effective plastic strain)
and then the same result is obtained using
\[ t p_{ij} = \frac{2}{3} t\sigma_y \left( \frac{d^t \sigma_y}{d^t \varepsilon^p} \frac{d^t \bar{\varepsilon}^p}{d^t \bar{\varepsilon}^p} \right) \]

Next consider \( t q_{ij} \):
\[
\begin{align*}
t q_{ij} &= \left. \frac{\partial^t F}{\partial^t \sigma_{ij}} \right|_{\text{fixed} \ t q_{ij}} = \frac{\partial}{\partial^t \sigma_{ij}} \left( \frac{1}{2} t s_{kl} t s_{kl} - \frac{1}{3} t\sigma_y^2 \right) \\
&= t s_{kl} \frac{\partial^t s_{kl}}{\partial^t \sigma_{ij}} = t s_{kl} \frac{\partial}{\partial^t \sigma_{ij}} \left( t\sigma_{kl} - \frac{t\sigma_{mm}}{3} \delta_{k\ell} \right) \\
&= t s_{kl} \left( \delta_{lk} \delta_{\ell} - \frac{\delta_{ij} \delta_{k\ell}}{3} \right) \\
&= t s_{ij} \text{ (note that } t s_{kl} \delta_{k\ell} = t s_{kk} = 0) 
\end{align*}
\]
We can now evaluate $C^{EP}$:

$$C^{EP} = \frac{E}{1 + \nu} \begin{bmatrix}
\delta e_{11} & \delta e_{22} & 2\delta e_{12} \\
(1 - \nu)\beta(s_{11}) & (1 - \nu)\beta(s_{22}) & (1 - \nu)\beta(s_{12}) \\
\delta e_{12} & \delta e_{12} & \delta e_{12}
\end{bmatrix}$$

where $\beta = \frac{3}{2} \frac{1}{\sigma_y} \left( \frac{1}{1 + \frac{2}{3} \frac{E}{E_T} \frac{1 + \nu}{E}} \right)$

Evaluation of the stresses at time $t + \Delta t$:

$$t + \Delta t \sigma = t \sigma + \int_t^{t + \Delta t} \sigma \, d\sigma$$

$$= t \sigma + \int_{\sigma}^{t + \Delta t \sigma} C^{EP} \, d\sigma$$

The stress integration must be performed at each Gauss integration point.
We can approximate the evaluation of this integral using the Euler forward method.

- Without subincrementation:

\[
\int_{t}^{t+\Delta t} C^{EP} \, dt \approx C^{EP} \bigg|_{t}^{t+\Delta t} \Delta e
\]

- With \( n \) subincrements:

\[
\int_{t}^{t+\Delta t} C^{EP} \, dt \approx C^{EP} \bigg|_{t}^{t+(n-1)\Delta t} \frac{\Delta e}{n} \\
+ C^{EP} \bigg|_{t+(n-1)\Delta t}^{t+n\Delta t} \frac{\Delta e}{n} \\
+ \ldots
\]
Summary of the procedure used to calculate the total stresses at time \( t+\Delta t \).

Given:

- \( \text{STRAIN} \) = Total strains at time \( t+\Delta t \)
- \( \text{SIG} \) = Total stresses at time \( t \)
- \( \text{EPS} \) = Total strains at time \( t \)

(a) Calculate the strain increment

\[ \text{DELEPS} = \text{STRAIN} - \text{EPS} \]
(b) Calculate the stress increment \( \Delta \sigma \), assuming elastic behavior:
\[
\Delta \sigma = C^E \times \Delta \epsilon
\]

(c) Calculate \( \tau \), assuming elastic behavior:
\[
\tau = \sigma + \Delta \sigma
\]

(d) With \( \tau \) as the state of stress, calculate the value of the yield function \( F \).

(e) If \( F(\tau) \leq 0 \), the strain increment is elastic. In this case, \( \tau \) is correct; we return.

(f) If the previous state of stress was plastic, set \( R \) to zero and go to (g). Otherwise, there is a transition from elastic to plastic and \( R \) (the portion of incremental strain taken elastically) has to be determined. \( R \) is determined from
\[
F(\sigma + R \times \Delta \epsilon) = 0
\]

since \( F = 0 \) signals the initiation of yielding.
(g) Redefine TAU as the stress at start of yield

\[ TAU = SIG + RATIO \times DELSIG \]

and calculate the elastic-plastic strain increment

\[ DEPS = (1 - RATIO) \times DELEPS \]

(h) Divide DEPS into subincrements DDEPS and calculate

\[ TAU \leftarrow TAU + \xi_{EP} \times DDEPS \]

for all elastic-plastic strain subincrements.
Plane strain punch problem

Finite element model of punch problem
Slide 17-3

Solution of Boussinesq problem—2 pt. integration

Slide 17-4

Solution of Boussinesq problem—3 pt. Integration
Load-displacement curves for punch problem
Limit load calculations:

- Plate is elasto-plastic.

Elasto-plastic analysis:

Material properties (steel)

- This is an idealization, probably inaccurate for large strain conditions ($e > 2\%$).
TIME = 0
LOAD = 0.0 MPA

Computer Animation
Plate with hole

TIME = 41
LOAD = 512.5 MPA

TIME = 52
LOAD = 550.0 MPA
Resource: Finite Element Procedures for Solids and Structures
Klaus-Jürgen Bathe

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