Topic 8

The Two-Noded Truss Element—Updated Lagrangian Formulation

Contents:

- Derivation of updated Lagrangian truss element displacement and strain-displacement matrices from continuum mechanics equations
- Assumption of large displacements and rotations but small strains
- Physical explanation of the matrices obtained directly by application of the principle of virtual work
- Effect of geometric (nonlinear strain) stiffness matrix
- Example analysis: Prestressed cable

Textbook: Section 6.3.1
Examples: 6.15, 6.16
TRUSS ELEMENT DERIVATION

A truss element is a structural member which incorporates the following assumptions:

- Stresses are transmitted only in the direction normal to the cross-section.
- The stress is constant over the cross-section.
- The cross-sectional area remains constant during deformations.

We consider the large rotation–small strain finite element formulation for a straight truss element with constant cross-sectional area.

Elastic material with Young's modulus E
Cross-sectional area A
Element lies in the $x_1 - x_2$ plane and is initially aligned with the $x_1$ axis.
The deformations of the element are specified by the displacements of its nodes:

Our goal is to determine the element deformations at time $t + \Delta t$.

Updated Lagrangian formulation:

The derivation is simplified if we consider a coordinate system aligned with the truss element at time $t$. 
Written in the rotated coordinate system, the equation of the principle of virtual work is

\[ \int_V t^+ \Delta t \hat{S}_{ij} \delta t^+ \Delta t \hat{\varepsilon}_{ij} \, t \, dV = t^+ \Delta t \bar{R} \]

As we recall, this may be linearized to obtain

\[ \int_V t \hat{C}_{ijrs} t \hat{\varepsilon}_{rs} \delta t \hat{\varepsilon}_{ij} \, t \, dV + \int_V t \hat{T}_{ij} \delta t \hat{\eta}_{ij} \, t \, dV \]

\[ = t^+ \Delta t \bar{R} - \int_V t \hat{T}_{ij} \delta t \hat{\varepsilon}_{ij} \, t \, dV \]

Because the only non-zero stress component is \( t \hat{T}_{11} \), the linearized equation of motion simplifies to

\[ \int_V t \hat{C}_{1111} t \hat{\varepsilon}_{11} \delta t \hat{\varepsilon}_{11} \, t \, dV + \int_V t \hat{T}_{11} \delta t \hat{\eta}_{11} \, t \, dV \]

\[ = t^+ \Delta t \bar{R} - \int_V t \hat{T}_{11} \delta t \hat{\varepsilon}_{11} \, t \, dV \]

Notice that we need only consider one component of the strain tensor.
We also notice that:

\[ \dot{C}_{1111} = E \]

\[ \dot{\tau}_{11} = \frac{tP}{A} \]

\[ \dot{V} = AL \]

The stress and strain states are constant along the truss.

Hence the equation of motion becomes

\[
(EA) \dot{\varepsilon}_{11} \delta t \dot{e}_{11} L + tP \delta t \eta_{11} L = t + \Delta t \dot{R} - tP \delta t \dot{e}_{11} L
\]

To proceed, we must express the strain increments in terms of the (rotated) displacement increments:

\[ \dot{\varepsilon}_{11} = B_L \dot{u}, \]

\[ \delta t \eta_{11} = (\delta \dot{u}^T B_{NL}^T)(B_{NL} \dot{u}) \]

where

\[
\dot{u} = \begin{bmatrix} \dot{u}_1^1 \\ \dot{u}_2^1 \\ \dot{u}_1^2 \\ \dot{u}_2^2 \end{bmatrix}
\]

This form is analogous to the form used in the two-dimensional element formulation.
Since $\tilde{\varepsilon}_{11} = \dot{u}_{1,1} + \frac{1}{2} ((\ddot{u}_{1,1})^2 + (\ddot{u}_{2,1})^2)$,
we recognize

$\tilde{\eta}_{11} = \dot{u}_{1,1}$

and

$\delta_t \tilde{\eta}_{11} = \delta_t \dot{u}_{1,1} = \delta_t \ddot{u}_{1,1} + \delta_t \ddot{u}_{2,1}$

$= [\delta_t \ddot{u}_{1,1} \ \delta_t \ddot{u}_{2,1}] \begin{bmatrix} \ddot{u}_{1,1} \\ \ddot{u}_{2,1} \end{bmatrix}$

matrix form

We can now write the displacement derivatives in terms of the displacements (this is simple because all quantities are constant along the truss). For example,

$\dot{u}_{1,1} = \frac{\partial \tilde{u}_1}{\partial x_1} = \frac{\Delta \tilde{u}_1}{\Delta x_1} = \frac{\dot{u}_1^2 - \tilde{u}_1^2}{L}$

Hence we obtain

$\begin{bmatrix} \dot{u}_{1,1} \\ \dot{u}_{2,1} \end{bmatrix} = \frac{1}{L} \begin{bmatrix} -1 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} \tilde{u}_1 \\ \tilde{u}_2 \\ \tilde{u}_1 \\ \tilde{u}_2 \end{bmatrix}$
and

\[ \tilde{\varepsilon}_{11} = \left( \begin{bmatrix} 1 & -1 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix} \right) \hat{u} \]

\[ \delta_{1}\hat{\eta}_{11} = \delta \hat{u}^T \left( \begin{bmatrix} 1 & -1 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix} \right) \left( \begin{bmatrix} 1 & -1 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix} \right)^T \hat{u} \]

Using these expressions,

\[ (EA) \tilde{\varepsilon}_{11} \delta_{1}\hat{\eta}_{11} L \]

\[ \delta \hat{u}^T \left( \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \right) \hat{u} \]

(setting successively each virtual nodal point displacement equal to unity)
\[ ^{tP} \delta(\tilde{\eta}_{11}) L \]

\[ \delta \hat{u}^T \left( ^{tP} \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix} \right) \hat{u} \]

and

\[ ^{tP} \delta(\tilde{\eta}_{11}) L \]

\[ \delta \hat{u}^T \left( ^{tP} \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \right) \]
We have now obtained the required element matrices, expressed in the coordinate system aligned with the truss at time $t$.

To determine the element matrices in the stationary global coordinate system, we must express the rotated displacement increments $\hat{u}$ in terms of the unrotated displacement increments $\tilde{u}$.

We can show that

$$
\begin{bmatrix}
\hat{u}_1 \\
\hat{u}_2
\end{bmatrix} =
\begin{bmatrix}
\cos^t \theta & \sin^t \theta \\
-sin^t \theta & \cos^t \theta
\end{bmatrix}
\begin{bmatrix}
\tilde{u}_1 \\
\tilde{u}_2
\end{bmatrix}
$$

Hence

$$
\begin{bmatrix}
\hat{u}_1 \\
\hat{u}_2 \\
\hat{u}_1' \\
\hat{u}_2'
\end{bmatrix} =
\begin{bmatrix}
\cos^t \theta & \sin^t \theta & 0 & 0 \\
-sin^t \theta & \cos^t \theta & 0 & 0 \\
0 & 0 & \cos^t \theta & \sin^t \theta \\
0 & 0 & -\sin^t \theta & \cos^t \theta
\end{bmatrix}
\begin{bmatrix}
\tilde{u}_1 \\
\tilde{u}_2 \\
\tilde{u}_1' \\
\tilde{u}_2'
\end{bmatrix}
$$
Using this transformation in the equation of motion gives

\[ \delta \hat{\mathbf{u}}^T \mathbf{K}_L \hat{\mathbf{u}} \rightarrow \delta \hat{\mathbf{u}}^T \mathbf{T}^T \mathbf{K}_L \mathbf{T} \hat{\mathbf{u}} \]

\[ \delta \hat{\mathbf{u}}^T \mathbf{K}_{NL} \hat{\mathbf{u}} \rightarrow \delta \hat{\mathbf{u}}^T \mathbf{T}^T \mathbf{K}_{NL} \mathbf{T} \hat{\mathbf{u}} \]

\[ \delta \hat{\mathbf{u}}^T \mathbf{F} \rightarrow \delta \hat{\mathbf{u}}^T \mathbf{T}^T \mathbf{F} \]

Performing the indicated matrix multiplications gives

\[ \mathbf{K}_L = \frac{EA}{L} \begin{bmatrix}
\cos^2 \theta & (\cos^2 \theta)(\sin^2 \theta) & -(\cos^2 \theta)(\sin^2 \theta) \\
(\cos^2 \theta)(\sin^2 \theta) & -\cos^2 \theta & -\sin^2 \theta \\
\sin^2 \theta & (\sin^2 \theta)(\cos^2 \theta) & -\sin^2 \theta \\
\end{bmatrix}
\]

symmetric

\[ \begin{bmatrix}
(\cos^2 \theta) & (\cos^2 \theta)(\sin^2 \theta) \\
(\sin^2 \theta)(\cos^2 \theta) & (\cos^2 \theta)(\sin^2 \theta) \\
(\sin^2 \theta) & (\sin^2 \theta)(\cos^2 \theta) \\
\end{bmatrix}
\]
The vector $\mathbf{F}$ makes physical sense:

For node 2,

$$\mathbf{R} = \begin{bmatrix} \mathbf{R} \cos \theta \\ \mathbf{R} \sin \theta \end{bmatrix}$$

$$\mathbf{F} = \begin{bmatrix} \mathbf{P} \cos \theta \\ \mathbf{P} \sin \theta \end{bmatrix}$$

Hence, at equilibrium,

$$\mathbf{R} - \mathbf{F} = 0$$
We note that the $\mathbf{K}_{\text{NL}}$ matrix is unchanged by the coordinate transformation.

- The nonlinear strain increment is related only to the vector magnitude of the displacement increment.

$$\sqrt{(\Delta u_1)^2 + (\Delta u_2)^2} = \left(\sqrt{\left(\frac{\partial \tilde{u}_1}{\partial x_1}\right)^2 + \left(\frac{\partial \tilde{u}_2}{\partial x_1}\right)^2}\right)L$$

Physically, $\mathbf{K}_{\text{NL}}$ gives the required change in the externally applied nodal point forces when the truss is rotated. Consider only $\tilde{u}_2$ nonzero.

For small $\tilde{u}_2$, this gives a rotation about node 1.

Moment equilibrium:

$$(\Delta \mathbf{R})(L) = \mathbf{P}(\tilde{u}_2)$$

or

$$\Delta \mathbf{R} = \frac{\mathbf{P}}{L} \tilde{u}_2$$

entry (4,4) of $\mathbf{K}_{\text{NL}}$

For small $\tilde{u}$,

$$\mathbf{K}_{\text{NL}} \tilde{u} = \frac{\mathbf{P}}{L} \Delta \mathbf{R}$$

- Internal force $\mathbf{P}$

- $\Delta \mathbf{R}$

- $\mathbf{K}_{\text{NL}}$
Example: Prestressed cable

![Diagram of a prestressed cable](image)

Finite element model (using symmetry):

\[
\begin{align*}
\text{Applied load } & 2^t R \\
\text{Initial tension } & = 0^o P \\
\text{Length } & 2L \\
\text{Young's modulus } & E \\
\text{Area } & A
\end{align*}
\]

Using the U.L. formulation, we obtain

\[
\left( \frac{EA}{L} (\sin^4 \theta)^2 + \frac{P}{L} \right) u_2^2 = t^+ + \Delta t R - t^P \sin \theta
\]

Of particular interest is the configuration at time 0, when \( t^\theta = 0 \):

\[
\left( \frac{0^o P}{L} \right) u_2^2 = \Delta t R
\]

The undeformed cable stiffness is given solely by \( K_{NL} \).
The cable stiffens as load is applied:

\[ iK = \frac{EA}{L} (\sin^2 \theta) + \frac{P}{L} \]

\( iK_L \) increases as \( \theta \) increases (the truss provides axial stiffness as \( \theta \) increases).

As \( \theta \rightarrow 90^\circ \), the stiffness approaches \( \frac{EA}{L} \),

but constant \( L \) and \( A \) means here that only small values of \( \theta \) are permissible.

Using: \( L = 120 \text{ in} \), \( A = 1 \text{ in}^2 \),
\( E = 30 \times 10^6 \text{ psi} \), \( P = 1000 \text{ lbs} \)

we obtain

\[ \text{Applied force (lbs)} \]

\[ \text{Deflection (inches)} \]

Graph not shown in text.
We also show the stiffness matrix components as functions of the applied load:

\[ K = K_L + K_{NL} \]