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Solutions Manual for Electromechanical Dynamics

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SOLUTIONS MANUAL FOR

ELECTROMECANICAL DYNAMICS

Part III: Elastic and Fluid Media

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PREFACE TO: SOLUTIONS MANUAL TO

ELECTROMECHANICAL DYNAMICS, PART III:

ELASTIC AND FLUID MEDIA

This manual presents, in an informal format, solutions to the problems found at the ends of chapters in Part III of the book, Electromechanical Dynamics. It is intended as an aid for instructors, and in special circumstances, for use by students. A sufficient amount of explanatory material is included such that the solutions, together with problem statements, are in themselves a teaching aid. They are substantially as found in the records for the undergraduate and graduate courses 6.06, 6.526, and 6.527, as taught at Massachusetts Institute of Technology over a period of several years.

It is difficult to give proper credit to all of those who contributed to these solutions, because the individuals involved range over teaching assistants, instructors, and faculty who have taught the material over a period of more than four years. However, special thanks are due the authors, Professor J. R. Melcher and Professor H. H. Woodson, who gave me the opportunity and incentive to write this manual. This work has greatly increased the value of my graduate education, in addition to giving me the pleasure of working with these two men.

The manuscript was typed by Mrs. Evelyn M. Holmes, whom I especially thank for her sense of humor, advice, patience and expertise which has made this work possible.

Of most value during the course of this work was the understanding of my girl friend, then fiancée, and now my wife, Linda, in spite of the competition for time.

Markus Zahn

Cambridge, Massachusetts
October, 1969
**PROBLEM 11.1**

**Part a**

We add up all the volume force densities on the elastic material, and with the help of equation 11.1.4, we write Newton's law as

\[
\rho \frac{\partial^2 \delta_1}{\partial t^2} = \frac{\partial T_{11}}{\partial x_1} - \rho g
\]  

where we have taken \( \frac{\partial}{\partial x_2} = \frac{\partial}{\partial x_3} = 0 \). Since this is a static problem, we let \( \frac{\partial}{\partial t} = 0 \). Thus,

\[
\frac{\partial T_{11}}{\partial x_1} = \rho g.
\]

From 11.2.32, we obtain

\[
T_{11} = (2G + \lambda) \frac{\partial \delta_1}{\partial x_1}
\]

Therefore

\[
(2G + \lambda) \frac{\partial^2 \delta_1}{\partial x_1^2} = \rho g
\]

Solving for \( \delta_1 \), we obtain

\[
\delta_1 = \frac{\rho g}{2(2G+\lambda)} x_1^2 + C_1 x + C_2
\]

where \( C_1 \) and \( C_2 \) are arbitrary constants of integration, which can be evaluated by the boundary conditions

\[
\delta_1(0) = 0
\]

and

\[
T_{11}(L) = (2G + \lambda) \frac{\partial \delta_1}{\partial x_1}(L) = 0
\]

since \( x_1 = L \) is a free surface. Therefore, the solution is

\[
\delta_1 = \frac{\rho g x_1}{2(2G+\lambda)} [x_1 - 2L].
\]

**Part b**

Again applying 11.2.32
PROBLEM 11.1 (Continued)

\[ T_{11} = (2G + \lambda) \frac{\partial \delta_1}{\partial x_1} = \rho g [x_1 - L] \]

\[ T_{12} = T_{21} = 0 \]

\[ T_{13} = T_{31} = 0 \]

\[ T_{22} = \lambda \frac{\partial \delta_1}{\partial x_1} = \frac{\lambda g}{(2G + \lambda)} [x_1 - L] \quad (g) \]

\[ T_{33} = \lambda \frac{\partial \delta_1}{\partial x_1} = \frac{\lambda g}{(2G + \lambda)} [x_1 - L] \]

\[ T_{32} = T_{23} = 0 \]

\[
\bar{T} = \begin{bmatrix}
T_{11} & 0 & 0 \\
0 & T_{22} & 0 \\
0 & 0 & T_{33}
\end{bmatrix}
\]

(h)

PROBLEM 11.2

Since the electric force only acts on the surface at \( x_1 = -L \), the equation of motion for the elastic material \((-L \leq x_1 \leq 0)\) is from Eqs. (11.1.4) and (11.2.32),

\[ \rho \frac{\partial^2 \delta_1}{\partial t^2} = (2G + \lambda) \frac{\partial^2 \delta_1}{\partial x_1^2} \quad (a) \]

The boundary conditions are

\[ \delta_1 (0, t) = 0 \]

and

\[ M \frac{\partial^2 \delta_1 (-L, t)}{\partial t^2} = aD (2G + \lambda) \frac{\partial \delta_1}{\partial x_1} (-L, t) + f^e \quad (b) \]

\( f^e \) is the electric force in the \( x_1 \) direction at \( x_1 = -L \), and may be found by using the Maxwell Stress Tensor \( T_{ij} = \varepsilon E_i E_j - \frac{1}{2} \delta_{ij} \varepsilon E_k E_k \) to be (see Appendix G for discussion of stress tensor),

\[ f^e = - \frac{\varepsilon}{2} E^2 aD \]

with

\[ E = \frac{V_0 + V_1 \cos wt}{d + \delta_1 (-L, t)} \quad (c) \]
PROBLEM 11.2 (continued)

Expanding \( f^e \) to linear terms only, we obtain

\[
f^e = -\frac{\varepsilon aD}{2} \left[ \frac{v^2}{\epsilon^2} + \frac{2V_2}{\epsilon^1} \frac{\cos \omega t}{d^2} - \frac{2v^2}{d^3} \delta_1(-L,t) \right]
\]

(d)

We have neglected all second order products of small quantities.

Because of the constant bias \( V_0 \), and the sinusoidal nature of the perturbations, we assume solutions of the form

\[
\delta_1(x_1,t) = \delta_1(x_1) + \Re(\delta e^{j(\omega t-kx_1)})
\]

(e)

where

\[
\delta \ll \delta_1(x_1) \ll L
\]

The relationship between \( \omega \) and \( k \) is readily found by substituting (e) into (a), from which we obtain

\[
k = \pm \frac{\omega}{v_p} \quad \text{with} \quad v_p = \sqrt{\frac{2G+\lambda}{\rho}}
\]

(f)

We first solve for the equilibrium configuration which is time independent. Thus

\[
\frac{\partial^2 \delta_1(x_1)}{\partial x_1^2} = 0
\]

(g)

This implies

\[
\delta_1(x_1) = C_1 x_1 + C_2
\]

Because \( \delta_1(0) = 0 \), \( C_2 = 0 \).

From the boundary condition at \( x_1 = -L \) ((b) & (d))

\[
aD(2G+\lambda)C_1 - \frac{\varepsilon}{2} \frac{\partial^2}{\partial x_1^2} aD = 0
\]

(h)

Therefore

\[
\delta_1(x_1) = \pm \frac{\varepsilon}{2} \frac{v^2}{d^2(2G+\lambda)} x_1
\]

(i)

Note that \( \delta_1(x_1 = -L) \) is negative, as it should be.

For the time varying part of the solution, using (f) and the boundary condition

\[
\delta(0,t) = 0
\]
PROBLEM 11.2 (continued)

we can let the perturbation $\delta_1$ be of the form

$$\delta_1(x_1,t) = \text{Re} \hat{\delta} \sin kx_1 e^{i\omega t} \quad (j)$$

Substituting this assumed solution into (b) and using (d), we obtain

$$+ Mw^2 \hat{\delta} \sin kL = aD(2G+\lambda)k \hat{\delta} \cos kL \quad (k)$$

$$- \frac{\varepsilon aDV_0}{d^2} \hat{\delta} \sin kL$$

Solving for $\hat{\delta}$, we have

$$\hat{\delta} = - \frac{\varepsilon aDV_0}{d^2} \left[ Mw^2 \sin kL - aD(2G+\lambda)k \cos kL + \frac{\varepsilon aDV_0^2}{d^3} \sin kL \right]$$

Thus, because $\hat{\delta}$ has been shown to be real,

$$\delta_1(-L,t) = \frac{\varepsilon aDV_0}{d^2(2G+\lambda)} \hat{\delta} \sin kL \cos \omega t \quad (m)$$

Part b

If $kL \ll 1$, we can approximate the sinusoidal part of (m) as

$$\delta_1(-L,t) = \frac{\varepsilon aDV_0}{d^2} \left[ Mw^2 - \frac{aD(2G+\lambda)}{L} + \frac{\varepsilon aDV_0^2}{d^3} \right] \cos \omega t \quad (n)$$

We recognize this as a force-displacement relation for a mass on the end of a spring.

Part c

We thus can model (n) as
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PROBLEM 11.2 (Continued)

where

\[ f = - \frac{\varepsilon aDV V_1 \cos \omega t}{d^2} \]

and

\[ K = aD(2G+\lambda) - \frac{\varepsilon aDV^2}{L} \frac{1}{d^3} \]

We see that the electrical force acts like a negative spring constant.

PROBLEM 11.3

Part a

From (11.1.4), we have the equation of motion in the \( x_2 \) direction as

\[ \rho \frac{\partial^2 \delta_2}{\partial t^2} = \frac{\partial T_{21}}{\partial x_1} \]  \hspace{1cm} (a)

From (11.2.32),

\[ T_{21} = G \begin{bmatrix} \delta_2 \\ \delta_2 \end{bmatrix} \frac{\partial}{\partial x_1} \]  \hspace{1cm} (b)

Therefore, substituting (b) into (a), we obtain an equation for \( \delta_2 \)

\[ \rho \frac{\partial^2 \delta_2}{\partial t^2} = G \frac{\partial^2 \delta_2}{\partial x_1^2} \]  \hspace{1cm} (c)

We assume solutions of the form

\[ \delta_2 = \text{Re} \left[ \hat{\delta}_2 e^{j(wt-kx_1)} \right] \]  \hspace{1cm} (d)

where from (c) we obtain

\[ k = \pm \frac{w}{v_p} \quad v_p^2 = \frac{G}{\rho} \]

Thus we let

\[ \delta_2 = \text{Re} \left[ \delta_a e^{j(wt-kx_1)} + \delta_b e^{j(wt+kx_1)} \right] \]  \hspace{1cm} (e)

with \( k = \frac{w}{v_p} \)

The boundary conditions are

\[ \delta_2(k,t) = \delta_o e^{j\omega t} \]  \hspace{1cm} (f)
and
\[ \frac{\partial \delta_2}{\partial x_1} \bigg|_{x_1=0} = 0 \] (g)
since the surface at \( x_1 = 0 \) is free.

Therefore
\[ \delta_a e^{-jk\ell} + \delta_b e^{jk\ell} = \delta_o \] (h)
and
\[ -jk \delta_a + jk \delta_b = 0 \] (i)
Solving, we obtain
\[ \delta_a = \delta_b = \frac{\delta_o}{2\cos k\ell} \] (j)
Therefore
\[ \delta_2(x_1,t) = \text{Re} \left[ \frac{\delta_o}{\cos k\ell} \cos k_1 e^{j\omega t} \right] = \frac{\delta_o}{\cos k\ell} \cos k_1 \cos \omega t \] (k)
and
\[ T_{21}(x_1,t) = -\text{Re} \left[ \frac{Gk \delta_o}{\cos k\ell} \sin k_1 e^{j\omega t} \right] \] (l)
\[ = -\frac{Gk \delta_o}{\cos k\ell} \sin k_1 \cos \omega t \]

Part b
In the limit as \( \omega \) gets small
\[ \delta_2(x_1,t) \rightarrow \text{Re}[\delta_o e^{j\omega t}] \] (m)
In this limit, \( \delta_2 \) varies everywhere in phase with the source. The slab of elastic material moves as a rigid body. Note from (l) that the force per unit area at \( x_1 = \ell \) required to set the slab into motion is \( T_{21}(\ell,t) = \rho \ell \frac{d^2}{dt^2}(\delta_o \cos \omega t) \) or the mass \( x_2 - x_3 \) area times the rigid body acceleration.

Part c
The slab can resonate if we can have a finite displacement, even as \( \delta_o \to 0 \). This can happen if the denominator of (k) vanishes
\[ \cos k\ell = 0 \] (n)
or
\[ \omega = \frac{(2n+1)\pi}{2\ell} \quad n = 0,1,2,\ldots \] (o)
PROBLEM 11.3 (continued)
The lowest frequency is for \( n = 0 \)

\[
\omega_{\text{low}} = \frac{\pi v P}{2\mu}
\]

PROBLEM 11.4

Part a

We have that

\[
T_i = T_{ij} n_j = \alpha \delta_{ij} n_j
\]

It is given that the \( T_{ij} \) are known, thus the above equation may be written as three scalar equations \((T_{ij} - \alpha \delta_{ij}) n_j = 0\), or:

\[
\begin{align*}
(T_{11} - \alpha)n_1 + T_{12}n_2 + T_{13}n_3 &= 0 \\
T_{21}n_1 + (T_{22} - \alpha)n_2 + T_{23}n_3 &= 0 \\
T_{31}n_1 + T_{32}n_2 + (T_{33} - \alpha)n_3 &= 0
\end{align*}
\]

Part b

The solution for these homogeneous equations requires that the determinant of the coefficients of the \( n_i \)'s equal zero.

Thus

\[
(T_{11} - \alpha)[(T_{22} - \alpha)(T_{33} - \alpha) - (T_{23})^2] - T_{12}[T_{12}(T_{33} - \alpha) - T_{13}T_{23}] + T_{13}[T_{12}T_{23} - T_{13}(T_{22} - \alpha)] = 0
\]

where we have used the fact that 

\[
T_{ij} = T_{ji}
\]

Since the \( T_{ij} \) are known, this equation can be solved for \( \alpha \).

Part c

Consider \( T_{12} = T_{21} = T_0 \), with all other components equal to zero. The determinant of coefficients then reduces to

\[
- \alpha^3 + T_0^2 \alpha = 0
\]

for which

\[
\alpha = 0
\]

or

\[
\alpha = \pm T_0
\]

The \( \alpha = 0 \) solution indicates that with the normal in the \( x_3 \) direction, there is no normal stress. The \( \alpha = \pm T_0 \) solution implies that there are two surfaces where the net traction is purely normal with stresses \( \pm T_0 \), respectively, as
PROBLEM 11.4 (continued)

found in example 11.2.1. Note that the normal to the surface for which the shear stress is zero can be found from (a), since \( \alpha \) is known, and it is known that \( |n| = 1 \).

PROBLEM 11.5

From Eqs. 11.2.25 - 11.2.28, we have

\[
\begin{align*}
e_{11} &= \frac{1}{E} [T_{11} - \nu(T_{22} + T_{33})] \\
e_{22} &= \frac{1}{E} [T_{22} - \nu(T_{33} + T_{11})] \\
e_{33} &= \frac{1}{E} [T_{33} - \nu(T_{11} + T_{22})]
\end{align*}
\]

and

\[
e_{ij} = \frac{T_{ij}}{2G} \quad i \neq j
\]

These relations must still hold in a primed coordinate system, where we can use the transformations

\[
T'_{ij} = a_{ik}a_{j\ell}T_{k\ell}
\]

and

\[
e'_{ij} = a_{ik}a_{j\ell}e_{k\ell}
\]

For an example, we look at \( e'_{11} \)

\[
e'_{11} = a_{1k}a_{1\ell}e_{k\ell} = \frac{1}{E} [T'_{11} - \nu(T'_{22} + T'_{33})]
\]

This may be rewritten as

\[
a_{1k}a_{1\ell}e_{k\ell} = \frac{1}{E} [(1 + \nu)a_{1k}a_{1\ell}T_{k\ell} - \nu \delta_{k\ell}T_{k\ell}]
\]

where we have used the relation from Eq. (8.2.23), page 410 or 439.

\[
a_{pr}a_{ps} = \delta_{ps}
\]

Consider some values of \( k \) and \( \ell \) where \( k \neq \ell \).

Then, from the stress-strain relation in the unprimed frame,

\[
a_{1k}a_{1\ell}e_{k\ell} = a_{1k}a_{1\ell} \frac{T_{k\ell}}{2G} = \frac{a_{1k}a_{1\ell}}{E} \frac{(1 + \nu)T_{k\ell}}{2G}
\]

Thus

\[
\frac{1}{2G} = \frac{1 + \nu}{E}
\]

or

\[ E = 2G(1 + \nu) \]

which agrees with Eq. (g) of example 11.2.1.
PROBLEM 11.6
Part a

Following the analysis in Eqs. 11.4.16 - 11.4.26, the equation of motion for the bar is

\[ \frac{\partial^2 \xi}{\partial t^2} + \frac{E b^2}{3 \rho} \frac{\partial^4 \xi}{\partial x_1^4} = 0 \]  

(a)

where \( \xi \) measures the bar displacement in the \( x_2 \) direction, \( T_2 \) in Eq. 11.4.26 = 0 as the surfaces at \( x_2 = \pm b \) are free. The boundary conditions for this problem are that at \( x_1 = 0 \) and at \( x_1 = L \)

\[ T_{21} = 0 \quad \text{and} \quad T_{11} = 0 \]  

(b)

as the ends are free.

We assume solutions of the form

\[ \xi = \Re \hat{\xi}(x)e^{j\omega t} \]  

(c)

As in example 11.4.4, the solutions for \( \hat{\xi}(x) \) are

\[ \hat{\xi}(x) = A \sin \alpha x_1 + B \cos \alpha x_1 + C \sinh \alpha x_1 + D \cosh \alpha x_1 \]  

(d)

with

\[ \alpha = \left[ \omega^2 \left( \frac{3 \rho}{E b^2} \right) \right]^{1/4} \]

Now, from Eqs. 11.4.18 and 11.4.21,

\[ T_{21} = \frac{(x_2^2 - b^2)E}{2} \frac{\partial^3 \xi}{\partial x_1^3} \]  

(e)

which implies

\[ \frac{\partial^3 \xi}{\partial x_1^3} = 0 \]  

(f)

at \( x_1 = 0, x_1 = L \)

and

\[ T_{11} = -x_2 E \frac{\partial^2 \xi}{\partial x_1^2} \]  

(g)

which implies

\[ \frac{\partial^2 \xi}{\partial x_1^2} = 0 \]  

(h)

at \( x_1 = 0 \) and \( x_1 = L \)
PROBLEM 11.6 (continued)

With these relations, the boundary conditions require that

\[-A + C = 0\]
\[-A \cos \alpha L + B \sin \alpha L + C \cosh \alpha L + D \sinh \alpha L = 0\]
\[-B + D = 0\]
\[-A \sin \alpha L - B \cos \alpha L + C \sinh \alpha L + D \cosh \alpha L = 0\]

The solution to this set of homogeneous equations requires that the determinant of the coefficients of A, B, C, and D equal zero. Performing this operation, we obtain

\[
\cos \alpha L \cosh \alpha L = 1
\]

Thus,

\[
\beta = \alpha L = \left( \frac{\omega^2}{2} \frac{3\rho}{Eb^2} \right)^{1/4} L
\]

Part b

The roots of \(\cos \beta = -\frac{1}{\cosh \beta}\) follow from the figure.

Note from the figure that the roots \(\alpha L\) are essentially the roots \(3\pi/2, 5\pi/2, \ldots\) of \(\cos \alpha L = 0\).
PROBLEM 11.6 (continued)

Part c

It follows from (i) that the eigenfunction is

\[ \hat{\xi} = A'[(\sin \alpha x_1 + \sinh \alpha x_1)(\sin \alpha L + \sinh \alpha L) + (\cos \alpha L - \cosh \alpha L)(\cos \alpha x_1 + \cosh \alpha x_1)] \]

where \( A' \) is an arbitrary amplitude. This expression is found by taking one of the constants \( A \dots D \) as known, and solving for the others. Then, (d) gives the required dependence on \( x_1 \) to within an arbitrary constant. A sketch of this function is shown in the figure.
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PROBLEM 11.7

As in problem 11.6, the equation of motion for the elastic beam is

\[ \frac{\partial^2 \xi}{\partial t^2} + \frac{E_b^2}{3\rho} \frac{\partial^4 \xi}{\partial x_1^4} = 0 \]  

(a)

The four boundary conditions for this problem are:

\[ \xi(x_1 = 0) = 0 \quad \xi(x_1 = L) = 0 \]

\[ \frac{\partial \xi}{\partial x_1}(0) = 0 \quad \frac{\partial \xi}{\partial x_1}(L) = 0 \]

(b)

We assume solutions of the form

\[ \xi(x_1, t) = \Re \hat{\xi}(x_1) e^{j\omega t} \]

and as in problem 11.6, the solutions for \( \hat{\xi}(x_1) \) are

\[ \hat{\xi}(x_1) = A \sin \alpha x_1 + B \cos \alpha x_1 + C \sinh \alpha x_1 + D \cosh \alpha x_1 \]

with \( \alpha = \left[ \omega^2 \frac{3\rho}{E_b^2} \right]^{1/4} \)

(c)

(d)

Applying the boundary conditions, we obtain

\[ B + D = 0 \]

\[ A \sin \alpha L + B \cos \alpha L + C \sinh \alpha L + D \cosh \alpha L = 0 \]

\[ A + C = 0 \]

\[ A \cos \alpha L - B \sin \alpha L + C \cosh \alpha L + D \sinh \alpha L = 0 \]

(e)

The solution to this set of homogeneous equations requires that the determinant of the coefficients of \( A, B, C, D, \) equal zero. Performing this operation, we obtain

\[ \cos \alpha L \cosh \alpha L = +1 \]

(f)

To solve for the natural frequencies, we must use a graphical procedure.
The first natural frequency is at about

$$\alpha L = \frac{3\pi}{2}$$

Thus

$$\omega^2 \left( \frac{3 \rho}{E b^2} \right) L^4 = \left( \frac{3\pi^6}{2} \right)$$

or

$$\omega = \frac{\left( \frac{3\pi}{2} \right)^2}{L^2} \left( \frac{E b^2}{3\rho} \right)^{1/2} \quad (g)$$

Part b

We are given that \( L = 0.5 \text{ m} \) and \( b = 5 \times 10^{-4} \text{ m} \)

From Table 9.1, Appendix G, the parameters for steel are:

- \( E \approx 2 \times 10^{11} \text{ N/m}^2 \)
- \( \rho \approx 7.75 \times 10^3 \text{ kg/m}^3 \)
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PROBLEM 11.7 (continued)

\[ \omega \approx 120 \text{ rad/sec.} \]

Then, \( f_1 = \frac{\omega}{2\pi} \approx 19 \text{ Hz.} \)

Part c

For the next higher resonance, \( \alpha L \approx \frac{5}{2} \pi \)

Therefore, \( f_2 = \left( \frac{5}{2} \right)^2 f_1 \approx 53 \text{ Hz.} \)

PROBLEM 11.8

Part a

As in Prob. 11.7, the equation of motion for the beam is

\[ \frac{\partial^2 \xi}{\partial t^2} + \frac{E b^2}{3p} \frac{\partial^4 \xi}{\partial x_1^4} = 0 \]  \hspace{1cm} (a)

At \( x_1 = L \), there is a free end, so the boundary conditions are:

\[ T_{11}(x_1 = L) = 0 \]
and

\[ T_{21}(x_1 = L) = 0 \]  \hspace{1cm} (b)

The boundary conditions at \( x_1 = 0 \) are

\[ M \left. \frac{\partial^2 \xi(0,t)}{\partial t^2} \right|_{x_1=0} = \int (T_{21})_{x_1=0} D \, dx_2 + \vec{F} + \vec{F}_0 \]  \hspace{1cm} (c)

and

\[ \delta_1(x_1 = 0) = 0 \]  \hspace{1cm} (d)

The \( \vec{H} \) field in the air gap and in the plunger is

\[ \vec{H} = \frac{N_i}{D} \vec{I}_1 \]  \hspace{1cm} (e)

Using the Maxwell stress tensor

\[ \vec{f}^e = -\frac{(\mu - \mu_0)}{2} \left( \frac{N_1^2 I_2^2}{D^2} \right) D^2 \vec{I}_2 = -\frac{N_1^2 I_2^2}{2} (\mu - \mu_0) \vec{I}_2 \]  \hspace{1cm} (f)

with \( I_1 = I_0 + i_1 \cos \omega t = I_0 + \text{Re} i_1 e^{j\omega t} \)
PROBLEM 11.8 (continued)

We linearize $\vec{F}$ to obtain

$$\vec{F} = -\frac{N^2}{2} \frac{\nu}{\nu_0} [I_0^2 + 2I_0 I_1 \cos \omega t]\vec{I}_2$$  \hspace{1cm} (g)

For equilibrium

$$\frac{F_o}{N} - \frac{N^2}{2} \frac{\nu}{\nu_0} I_o^2 \vec{I}_2 = 0$$

Thus

$$\frac{F_o}{N} = \frac{N^2}{2} \frac{\nu}{\nu_0} I_o^2 \vec{I}_2$$  \hspace{1cm} (h)

Part b

We write the solution to Eq. (a) in the form

$$\xi(x_1, t) = \text{Re} \hat{\xi}(x_1) e^{j\omega t}$$

where, from example 11.4.4

$$\hat{\xi}(x_1) = A_1 \sin \alpha x_1 + A_2 \cos \alpha x_1 + A_3 \sinh \alpha x_1 + A_4 \cosh \alpha x_1$$  \hspace{1cm} (i)

with

$$\alpha = \left[ \omega^2 \left( \frac{3\rho}{E\beta^2} \right) \right]^{1/4}$$

Now, from Eqs. 11.4.6 and 11.4.16

$$T_{11}(x=L) = E \frac{\delta \xi}{\delta x_1} = -E \frac{\partial^2 \xi}{\partial x_1^2} = 0$$  \hspace{1cm} (j)

Thus

$$\frac{\partial^2 \xi}{\partial x_1^2} (x_1 = L) = 0$$

From Eq. 11.4.21

$$T_{21} = \frac{\frac{x_2^2 - b^2}{2}}{E} \frac{\partial^3 \xi}{\partial x_1^3}$$  \hspace{1cm} (k)

and from Eq. 11.4.16

$$\delta_1(x_1 = 0) = -x_2 \left( \frac{\partial^3 \xi}{\partial x_1^3} \right)_{x_1=0} = 0$$  \hspace{1cm} (l)

Thus

$$\left( \frac{\partial^3 \xi}{\partial x_1^3} \right)_{x_1=0} = 0$$
PROBLEM 11.8 (continued)

Applying the boundary conditions from Eqs. (b), (c), (d) to our solution of Eq. (i), we obtain the four equations

\[ A_1 + A_3 = 0 \]
\[ -A_1 \sin \alpha l - A_2 \cos \alpha l + A_3 \sinh \alpha l + A_4 \cosh \alpha l = 0 \]
\[ -A_1 \cos \alpha l + A_2 \sin \alpha l + A_3 \cosh \alpha l + A_4 \sinh \alpha l = 0 \]
\[ -\frac{2}{3} a^3 b^3 EDA_1 + M \omega^2 A_2 + \frac{2}{3} a^3 b^3 EDA_3 + M \omega^2 A_4 = +N^2 I_o i_1 (\mu - \mu_o) \]

Now
\[ v = \frac{d\lambda}{dt} = \frac{d}{dt} \left( \frac{N^2 I_o i_1}{D} \left[ \mu_o \xi(0) + \mu \xi(0) \right] \right) \]

or
\[ \hat{v} = -N^2 I_o (\mu - \mu_o) j\omega (A_2 + A_4) + N^2 I_o i_1 \mu D j\omega \]

We solve Eqs. (m) for \( A_2 \) and \( A_4 \) using Cramer's rule to obtain
\[ A_2 = \frac{N^2 I_o i_1 (\mu - \mu_o) (-1 + \sin \alpha l \sinh \alpha l - \cos \alpha l \cosh \alpha l)}{-2M\omega^2 (1 + \cos \alpha l \cosh \alpha l) + \frac{4}{3} (ab)^3 ED (\cos \alpha l \sinh \alpha l + \sin \alpha l \cosh \alpha l)} \]

and
\[ A_4 = \frac{N^2 I_o i_1 (\mu - \mu_o) (-1 - \cos \alpha l \cosh \alpha l - \sin \alpha l \sinh \alpha l)}{-2M\omega^2 (1 + \cos \alpha l \cosh \alpha l) + \frac{4}{3} (ab)^3 ED (\cos \alpha l \sinh \alpha l + \sin \alpha l \cosh \alpha l)} \]

Thus
\[ Z(j\omega) = \frac{\hat{V}(j\omega)}{i_1} = \frac{\left[ N^2 I_o (\mu - \mu_o) \right]^2 j\omega (+2 + 2 \cos \alpha l \cosh \alpha l)}{-2M\omega^2 (1 + \cos \alpha l \cosh \alpha l) + \frac{4}{3} (ab)^3 ED (\cos \alpha l \sinh \alpha l + \sin \alpha l \cosh \alpha l)} \]

\[ + N^2 \mu D j\omega \]

Part c

\[ Z(j\omega) \text{ has poles when} \]
\[ +2M\omega^2 (1 + \cos \alpha l \cosh \alpha l) = \frac{4}{3} (ab)^3 ED (\cos \alpha l \sinh \alpha l + \sin \alpha l \cosh \alpha l) \]
PROBLEM 11.9
Part a

The flux above and below the beam must remain constant. Therefore, the \( H \) field above is

\[
\bar{H}_a = \frac{H_0 (a-b)}{(a-b-\xi)^T}
\]

and the \( H \) field below is

\[
\bar{H}_b = \frac{H_0 (a-b)}{(a-b+\xi)}
\]

Using the Maxwell stress tensor, the magnetic force on the beam is

\[
T_2 = -\frac{\mu_0}{2} \left( \bar{H}_a^2 - \bar{H}_b^2 \right) = -\frac{\mu_0}{2} H_0^2 (a-b)^2 \left( + \frac{4 \xi}{(a-b)^3} \right)
\]

\[
\frac{2 \mu_0 H_0^2 \xi}{(a-b)}
\]

Thus, from Eq. 11.4.26, the equation of motion on the beam is

\[
\frac{3 \xi^2}{2 \xi^2 + \frac{\text{Eb}^2}{3 \rho} \frac{\partial^4 \xi}{\partial x_1^4} = - \frac{\mu_0 H_0^2 \xi}{(a-b) \rho}}
\]

Again, we let

\[
\xi(x_1,t) = \text{Re} \hat{\xi}(x_1) e^{j \omega t}
\]

with the boundary conditions

\[
\xi(x_1=0) = 0 \quad \xi(x_1=L) = 0
\]

\[
\delta_1(x_1=0) \quad \delta_1(x_1=L) = 0
\]

Since \( \delta_1 = -x_2 \frac{\partial \xi}{\partial x_1} \) from Eq. 11.4.16, this implies that:

\[
\frac{3 \xi}{\partial x_1} (x_1=0) = 0 \quad \text{and} \quad \frac{3 \xi}{\partial x_1} (x_1=L) = 0
\]

Substituting our assumed solution into the equation of motion, we have

\[
-\omega^2 \hat{\xi} = \text{Eb}^2 \frac{\partial^4 \hat{\xi}}{\partial x_1^4} + \frac{\mu_0 H_0^2 \hat{\xi}}{(a-b) \rho}
\]

Thus we see that our solutions are again of the form

\[
\hat{\xi}(x) = A \sin \alpha x + B \cos \alpha x + C \sinh \alpha x + D \cosh \alpha x
\]
where now
\[ \alpha = \left[ \frac{\omega^2}{\omega_0^2} - \frac{\mu_0 H_0^2}{(a-b)b\rho} \right]^{1/4} \]  

Since the boundary conditions for this problem are identical to that of problem 11.7, we can take the solutions from that problem, substituting the new value of \( \alpha \). From problem 11.7, the solution must satisfy

\[ \cos \alpha L \cosh \alpha L = 1 \]  

The first resonance occurs when

\[ \alpha L \approx \frac{3\pi}{2} \]

or

\[ \omega^2 = \left( \frac{3\pi}{2} \right)^2 \left( \frac{\mu_0 H_0^2}{3\rho} \right) L^4 + \frac{\mu_0 H_0^2}{(a-b)b\rho} \]  

Part c

The resonant frequencies are thus shifted upward due to the stiffening effect of the constant flux constraint.

Part d

We see that, no matter what the values of the system parameters \( \omega^2 > 0 \), so \( \omega \) will always be real, and thus stable. This is expected as the constant flux constraint imposes a force which opposes the motion.

PROBLEM 11.10

Part a

We choose a coordinate system as in Fig. 11.4.12, centered at the right end of the rod. Because \( \frac{d}{D} = \frac{1}{10} \), we can neglect fringing and consider the right end of the rod as a capacitor plate. Also, since \( \frac{D}{L} = \frac{1}{10} \), we can assume that the electrical force acts only at \( x_1 = 0 \). Thus, the boundary conditions at \( x_1 = 0 \) are

\[ - \int_{x_2}^{b} T_{21} D dx_2 + f^e = 0 \]  

where \( T_{21} = \frac{(x_2^2 - b^2)}{2} E \frac{3\epsilon_0}{3x^3} \) (Eq. 11.4.21)

since the electrical force, \( f^e \), must balance the shear stress \( T_{21} \) to keep the rod in equilibrium.
PROBLEM 11.10 (continued)

and

\[ T_{11}(0) = - x_2 E \frac{\partial^2 \xi}{\partial x_1^2}(0) = 0 \]  

(b)

since the end of the rod is free of normal stresses. At \( x_1 = -2 \), the rod is clamped so

\[ \xi(-2) = 0 \]  

(c)

and

\[ \delta_1(-2) = - x_2 \frac{\partial \xi}{\partial x_1}(-2) = 0 \]  

(d)

We use the Maxwell stress tensor to calculate the electrical force to be

\[ f_e = \frac{\varepsilon A}{2} \left[ \frac{(V_o + V_s)^2}{(d - \xi(0))^2} - \frac{(V_o - V_s)^2}{(d + \xi(0))^2} \right] \]  

(e)

The equation of motion of the beam is (example 11.4.4)

\[ \frac{\partial^2 \xi}{\partial t^2} + \frac{E h^2}{3 \rho} \frac{\partial^2 \xi}{\partial x_1^2} = 0 \]  

(f)

We write the solution to Eq. (f) in the form

\[ \xi(x,t) = \text{Re} \hat{\xi}(x)e^{j\omega t} \]  

(g)

where

\[ \hat{\xi}(x) = A_1 \sin \alpha x + A_2 \cos \alpha x + A_3 \sinh \alpha x + A_4 \cosh \alpha x \]

with

\[ \alpha = \left( \frac{\omega}{\sqrt{3 \rho E h^2}} \right) \]

Applying the four boundary conditions, Eqs. (a), (b), (c) and (d), we obtain the equations

\[- A_1 \sin \alpha L + A_2 \cos \alpha L + A_3 \sinh \alpha L + A_4 \cosh \alpha L = 0 \]

(h)

\[- A_1 \cos \alpha L + A_2 \sin \alpha L + A_3 \cosh \alpha L - A_4 \sinh \alpha L = 0 \]

\[- A_2 + A_4 = 0 \]

\[- \frac{2}{3} b^3 D E \alpha A_1 + \frac{2 E_0 A V^2}{d^2} A_2 + \frac{2}{3} b^3 D E \alpha A_3 + \frac{2 E_o A V^2}{d^3} A_4 = - \frac{2 E_o A V_s}{d^2} \]
PROBLEM 11.10 (continued)

Now \( i_s = \frac{dq_s}{dt} \) \hspace{1cm} (i)

where \( q_s = \frac{\varepsilon_o A}{d-\xi(0)} (V + v_s) + \frac{\varepsilon_o A(v - V_o)}{d + \xi(0)} \)

\( \xi = \frac{2\varepsilon_o AV_s}{d} + \frac{2\varepsilon_o AV}{d} \xi(0) \) \hspace{1cm} (j)

Therefore

\[ i_s = j\omega \frac{2\varepsilon_o A}{d} \left[ \frac{\hat{V}}{s} + \frac{V_o}{d} \hat{\xi}(0) \right] \] \hspace{1cm} (k)

where \( \hat{\xi}(0) = A_2 + A_4 \)

We use Cramer's rule to solve Eqs. (h) for \( A_2 \) and \( A_4 \) to obtain:

\[ A_2 = A_4 = \frac{-\varepsilon_o AV_s}{d^2} \left[ \cos \alpha \text{ sinh } \alpha - \sin \alpha \text{ cosh } \alpha \right] \]

\[ A_2 = A_4 = \frac{2}{3} b^2 \varepsilon_o \alpha^2 \left( 1 + \cos \alpha \text{ cosh } \alpha \right) + \frac{2\varepsilon_o AV}{d^3} \left( \cos \alpha \text{ sinh } \alpha - \sin \alpha \text{ cosh } \alpha \right) \]

Thus, from Eq. (k) we obtain

\[ Z(j\omega) = \frac{d}{j\omega 2\varepsilon_o A} \left[ 1 + \frac{3\varepsilon_o AV^2}{d^3 (\alpha \beta)^3 ED} \left( \cos \alpha \text{ sinh } \alpha - \sin \alpha \text{ cosh } \alpha \right) \right] \]

\hspace{1cm} (m)

**Part b**

We define a function \( g(\alpha \beta) \) such that Eq. (m) has a zero when
PROBLEM 11.10 (Continued)

\[
(aL)^3 g(aL) = \frac{(1 + \cosh \alpha L \cos \alpha L)(\alpha L)^3}{\sin \alpha L \cosh \alpha L - \cos \alpha L \sinh \alpha L} = \frac{3L^3 V^2 AE_0}{D E b^3 d^3} (n)
\]

Substituting numerical values, we obtain

\[
\frac{3L^3 V^2 AE_0}{D E b^3 d^3} \approx \frac{3 \times 10^{-3} (10^2) 10^{-4} (8.85 \times 10^{-12})}{10^{-2} (2.2 \times 10^{11}) 10^{-9}} \approx 1.2 \times 10^{-3} (o)
\]

In Figure 1, we plot \((aL)^3 g(aL)\) as a function of \(aL\). We see that the solution to Eq. (n) first occurs when \((aL)^3 g(aL) \approx 0\). Thus, the solution is approximately

\[aL = 1.875\]

---

Figure 1
PROBLEM 11.10 (Continued)

From Eq. (g)

\[ \alpha \ell = \left[ \omega^2 \frac{3p}{E_b} \right] \quad \ell = 1.875 \]

Solving for \( \omega \), we obtain

\[ \omega \approx 1080 \text{ rad/sec.} \] (p)

Part c

The input impedance of a series LC circuit is

\[ Z(j\omega) = \frac{1 - LC\omega^2}{j\omega C} \] (q)

Thus the impedance has a zero when

\[ \omega_o = \frac{1}{LC} \] (r)

We let \( \omega = \omega_o + \Delta \omega \), and expand \( Z(j\omega) \) in a Taylor series around \( \omega_o \) to obtain

\[ Z(j\omega) \approx j \frac{2\Delta \omega}{C\omega_o^2} + 2j LC \Delta \omega \] (s)

(m) can be written in the form

\[ Z(j\omega) = \frac{1}{2j\omega C_o} [1 - f(\omega)] \quad \text{where } f(\omega_o) = 1 \]

\[ \quad \text{and } C_o = \frac{\varepsilon_o \Delta}{d} \] (t)

For small deviations around \( \omega_o \)

\[ Z(j\omega) \approx \frac{j}{2\omega C_o} \left. \frac{\partial f}{\partial \omega} \right|_{\omega_o} \Delta \omega \]

Thus, from (q), (r) (s) and (t), we obtain the relations

\[ 2L = \frac{1}{2\omega C_o} \left. \frac{3f}{\partial \omega} \right|_{\omega_o} \] (u)

and \[ C = \frac{1}{\omega_o^2 L} \] (v)

now \[ f(\omega) = \frac{K}{(\alpha \ell)^3 g(\alpha \ell)} \] (w)

where \[ K = \frac{3\varepsilon^3 \varepsilon_o A V^2}{d^3 (E_b)^3} = 1.2 \times 10^{-3} \]
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PROBLEM 11.10 (Continued)

and \( g(\alpha \ell) = \frac{1 + \cos \alpha \ell \cosh \alpha \ell}{\sin \alpha \ell \cosh \alpha \ell - \cos \alpha \ell \sinh \alpha \ell} \)

Thus, we can write

\[
\frac{df(\omega)}{d\omega} \bigg|_{\omega_0} = \left\{ \frac{d}{d(\alpha \ell)} \left[ \frac{K}{(\alpha \ell)^2 g(\alpha \ell)} \right] \frac{d(\alpha \ell)}{d\omega} \right\} \bigg|_{\omega_0} \tag{y}
\]

Now from (g),

\[
\frac{d(\alpha \ell)}{d\omega} \bigg|_{\omega_0} = \left( \frac{3\rho}{E_b} \right)^{1/4} \frac{K}{2\omega_0^{1/2}} \tag{z}
\]

and

\[
\frac{d}{d(\alpha \ell)} \left[ \frac{K}{(\alpha \ell)^3 g(\alpha \ell)} \right] \bigg|_{\omega_0} = -\frac{K}{(\alpha \ell)^3 g(\alpha \ell)} \left( \frac{d(\alpha \ell)}{d\omega} \right)^2 \frac{d(\alpha \ell)}{d\omega} \left[ (\alpha \ell)^3 g(\alpha \ell) \right] \bigg|_{\omega_0}
\]

\[
\frac{\alpha \ell}{K} - \frac{1}{K} \frac{d}{d(\alpha \ell)} \left[ (\alpha \ell)^3 g(\alpha \ell) \right] \bigg|_{\omega_0} \tag{aa}
\]

since at \( \omega = \omega_0 \)

\[
(\alpha \ell)^3 g(\alpha \ell) = K. \tag{bb}
\]

Continuing the differentiating in (aa), we finally obtain

\[
\frac{d}{d(\alpha \ell)} \left[ \frac{(\alpha \ell)^3 g(\alpha \ell)}{K - K} \right] \bigg|_{\omega_0} = -\frac{1}{K} \left[ g(\alpha \ell)^3 (\alpha \ell)^2 + (\alpha \ell)^3 \frac{d}{d(\alpha \ell)} g(\alpha \ell) \right] \bigg|_{\omega_0}
\]

\[
= -\frac{3}{\alpha \ell} \left| \frac{1}{K} \frac{d}{d(\alpha \ell)} g(\alpha \ell) \right|_{\omega_0} \tag{cc}
\]

Now

\[
\frac{d}{d(\alpha \ell)} g(\alpha \ell) = \frac{-\sin \alpha \ell \cosh \alpha \ell + \cos \alpha \ell \sinh \alpha \ell}{(\sin \alpha \ell \cosh \alpha \ell - \cos \alpha \ell \sinh \alpha \ell)}
\]

\[
= -1 - \frac{2g(\alpha \ell)}{(\sin \alpha \ell \sinh \alpha \ell)} \frac{\sin \alpha \ell \sinh \alpha \ell + \sin \alpha \ell \cosh \alpha \ell - \cos \alpha \ell \cosh \alpha \ell}{(\sin \alpha \ell \cosh \alpha \ell - \cos \alpha \ell \sinh \alpha \ell)} \frac{(\sin \alpha \ell \cosh \alpha \ell - \cos \alpha \ell \sinh \alpha \ell)}{(\sin \alpha \ell \cosh \alpha \ell - \cos \alpha \ell \sinh \alpha \ell)} \tag{dd}
\]

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PROBLEM 11.10 (Continued)

Substituting numerical values into the second term of (cc), we find it to have value much less than one at \( \omega = \omega_0 \).

Thus,

\[
\frac{d}{d(\alpha t)} g(\alpha t) \approx 1
\]

Thus, using (y), (z), (aa), (bb) and (dd), we have

\[
\frac{df}{d\omega} \Bigg|_{\omega = \omega_0} \approx \left( \frac{3\sqrt{E}}{Eb^2} \right) \frac{C}{2\omega_0^{3/2}} \left[ -\frac{3}{\omega_0^3} + \frac{(\alpha t)^3}{K} \right] \approx 4.8
\]

Thus, from (v) and (w)

\[
L \approx \frac{4.8 \times 10^{-3}}{4(1080)(8.85 \times 10^{-12})(10^{-6})} = 1.25 \times 10^9 \text{ henries}
\]

and

\[
C \approx \frac{1}{1.25 \times 10^9 (1080)^2} = 6.8 \times 10^{-16} \text{ farads}.
\]
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PROBLEM 11.11

From Eq. (11.4.29), the equation of motion is

\[ \frac{\partial^2 \delta}{\partial t^2} = \frac{G}{\partial x_1} \left( \frac{\partial^2 \delta}{\partial x_1^2} + \frac{\partial^2 \delta}{\partial x_2^2} \right) \]  

(a)

We let

\[ \delta = \Re \hat{\delta}(x_2) e^{i(\omega t - kx_1)} \]  

(b)

Substituting this assumed solution into the equation of motion, we obtain

\[ -\rho \omega^2 \hat{\delta} = G \left( -k^2 \hat{\delta} + \frac{\partial^2 \hat{\delta}}{\partial x_2^2} \right) \]  

(c)

or

\[ \frac{\partial^2 \hat{\delta}}{\partial x_2^2} + \left( \frac{\rho \omega^2}{G} - k^2 \right) \hat{\delta} = 0 \]  

(d)

If we let \( \beta^2 = \frac{\rho \omega^2}{G} - k^2 \)

(e)

the solutions for \( \hat{\delta} \) are:

\[ \hat{\delta}(x_2) = A \sin \beta x_2 + B \cos \beta x_2 \]  

(f)

The boundary conditions are

\[ \hat{\delta}(0) = 0 \quad \text{and} \quad \hat{\delta}(d) = 0 \]  

(g)

This implies that \( B = 0 \) and that \( \beta d = n\pi \).

Thus, the dispersion relation is

\[ \omega^2 \frac{G}{\rho} - k^2 = \left( \frac{n\pi}{d} \right)^2 \]  

(h)

Part b

The sketch of the dispersion relation is identical to that of Fig. 11.4.19. However, now the \( n=0 \) solution is trivial, as it implies that

\[ \hat{\delta}(x_2) = 0 \]

Thus, there is no principal mode of propagation.
PROBLEM 11.12

From Eq. (11.4.1), the equation of motion is

$$\rho \frac{\partial^2 \delta}{\partial t^2} = (2G + \lambda) \nabla (\nabla \cdot \delta) - G \nabla \times (\nabla \times \delta)$$  \hspace{1cm} (a)

We consider motions

$$\delta = \delta_\theta (r, z, t) \hat{e}_\theta$$ \hspace{1cm} (b)

Thus, the equation of motion reduces to

$$\rho \frac{\partial^2 \delta_\theta}{\partial t^2} - \nabla \left[ \nabla \frac{\partial^2 \delta_\theta}{\partial z^2} + \frac{1}{r} \frac{\partial}{\partial r} \left( \frac{3}{r} \frac{\partial \delta_\theta}{\partial r} \right) \right] = 0$$ \hspace{1cm} (c)

We assume solutions of the form

$$\delta_\theta (r, z, t) = \text{Re} \hat{\delta}(r)e^{i(\omega t - k z)}$$ \hspace{1cm} (d)

which, when substituted into the equation of motion, yields

$$\frac{\partial}{\partial r} \left[ \frac{1}{r} \frac{\partial}{\partial r} \hat{\delta}(r) \right] + \left( \frac{\rho \omega^2}{G} - k^2 \right) \hat{\delta}(r) = 0$$ \hspace{1cm} (e)

From page 207 of Ramo, Whinnery and Van Duze, we recognize solutions to this equation as

$$\hat{\delta}(r) = A J_1 \left[ \left( \frac{\rho \omega^2}{G} - k^2 \right)^{1/2} r \right] + B N_1 \left[ \left( \frac{\rho \omega^2}{G} - k^2 \right)^{1/2} r \right]$$ \hspace{1cm} (f)

On page 209 of this reference there are plots of the Bessel functions $J_1$ and $N_1$. We must have $B = 0$ as at $r = 0$, $N_1$ goes to $-\infty$. Now, at $r = R$

$$\hat{\delta}(R) = 0$$ \hspace{1cm} (g)

This implies that

$$J_1 \left[ \left( \frac{\rho \omega^2}{G} - k^2 \right)^{1/2} R \right] = 0$$ \hspace{1cm} (h)

If we denote $\alpha_1$ as the zeroes of $J_1$, i.e.

$$J_1 (\alpha_1) = 0$$

we have the dispersion relation as

$$\frac{\rho}{G} \omega^2 - k^2 = \frac{\alpha_1^2}{R^2}$$ \hspace{1cm} (i)
PROBLEM 12.1

Part a

Since we are in the steady state \( \frac{\partial}{\partial t} = 0 \), the total forces on the piston must sum to zero. Thus

\[
pLD + (f^e)_x = 0 \tag{a}
\]

where \((f^e)_x\) is the upwards vertical component of the electric force

\[
(f^e)_x = -\frac{\varepsilon_0 V_o^2}{2x^2} \quad LD
\]

Solving for the pressure \( p \), we obtain

\[
p = \frac{\varepsilon_0 V_o^2}{2x^2} \tag{c}
\]

Part b

Because \( \frac{d}{L} \ll 1 \), we approximate the velocity of the piston to be negligibly small. Then, applying Bernoulli's equation, Eq. (12.2.11) right below the piston and at the exit nozzle where the pressure is zero, we obtain

\[
\frac{1}{2} \rho V^2_p = \frac{\varepsilon_0 V_o^2}{2x^2} \tag{d}
\]

Solving for \( V_p \), we have

\[
V_p = \frac{V_o}{x} \sqrt{\frac{\varepsilon_0}{\rho}} \tag{e}
\]

Part c

The thrust \( T \) on the rocket is then

\[
T = \int \frac{dM}{\rho \, dt} = V^2 \rho dD = \frac{\varepsilon_0 V_o^2}{x^2} \quad dD \tag{f}
\]

PROBLEM 12.2

Part a

The forces on the movable piston must sum to zero. Thus

\[
pD - f^e = 0 \tag{a}
\]

where \( f^e \) is the component of electrical force normal to the piston in the direction of \( V \), and \( p \) is the pressure just to the right of the piston.

\[
f^e = \frac{\mu_o \, r^2 D}{2x} \tag{b}
\]
Therefore
\[ p = \frac{\mu_0 I^2}{2w} \quad (c) \]

Assuming that the velocity of the piston is negligible, we use Bernoulli's law, Eq. (12.2.11), just to the right of the piston and at the exit orifice where the pressure is zero, to write
\[ \frac{1}{2} \rho V^2 = p \quad (d) \]

or
\[ V = \frac{I}{\sqrt{\frac{\mu_0}{W}}} \rho \quad (e) \]

**Part b**

The thrust \( T \) is
\[ T = V \frac{dM}{dt} = V^2 \rho dW = \frac{\mu_0 I^2 d}{W} \quad (f) \]

**Part c**

For \( I = 10^3 A \)
- \( d = .1m \)
- \( w = .1m \)
- \( \rho = 10^3 \text{ kg/m}^3 \)

the exit velocity is
\[ V = 3.5 \times 10^{-2} \text{ m/sec.} \]

and the thrust is
\[ T = .126 \text{ newtons.} \]

Within the assumption that the fluid is incompressible, we would prefer a dense material, for although the thrust is independent of the fluid's density, the exhaust velocity would decrease with increasing density, and thus the rocket will work longer. Under these conditions, we would prefer water in our rocket, since it is much more dense than air.

**PROBLEM 12.3**

**Part a**

From the results of problem 12.2, we have that the pressure \( p \), acting just to the left of the piston, is
\[ p = \frac{\mu_0 I^2}{2w} \quad (a) \]

The exit velocity at each orifice is obtained by using Bernoulli's law just to the left of the piston and at either orifice, from which we obtain
PROBLEM 12.3 (Continued)

\[ v = \left( \frac{\mu_0}{\rho} \right)^{1/2} \frac{l}{w} \]  \hspace{1cm} (b)

at each orifice.

Part b

The thrust is

\[ T = 2V \frac{dM}{dt} = 2V^2 \rho dw \]  \hspace{1cm} (c)

\[ T = \frac{2 \mu I^2 d}{w} \]  \hspace{1cm} (d)

PROBLEM 12.4

Part a

In the steady state, we choose to integrate the momentum theorem, Eq. (12.1.29), around a rectangular surface, enclosing the system from \(-L < x < +L\).

\[ -\rho v_o^2 a + \rho \{v(L)\}^2 b = P_o a - P(L)b + F \]  \hspace{1cm} (a)

where \( F \) is the \( x \) component force per unit length which the walls exert on the fluid. We see that there is no \( x \) component of force from the upper wall, therefore \( F \) is the force purely from the lower wall.

In the steady state, conservation of mass, (Eq. 12.1.8), yields

\[ v(L) = v_o \frac{a}{b} \]  \hspace{1cm} (b)

Bernoulli's equation gives us

\[ \frac{1}{2} \rho v_o^2 + P_o = \frac{1}{2} \rho v_o^2 \frac{a^2}{b^2} + P(L) \]  \hspace{1cm} (c)

Solving (c) for \( P(L) \), and then substituting this result and that of (b) into (a), we finally obtain

\[ F = P_o (b-a) + \rho v_o^2 (-a + \frac{b + a^2}{2b} ) \]  \hspace{1cm} (d)

The problem asked for the force on the lower wall, which is just the negative of \( F \).

Thus

\[ F_{wall} = -P_o (b-a) - \rho v_o^2 (-a + \frac{a^2}{2b} + \frac{b}{2} ) \]  \hspace{1cm} (e)

PROBLEM 12.5

Part a

We recognize this problem to be analogous to a dielectric or high-permeability cylinder placed in a uniform electric or magnetic field. The solutions are then dipole fields. We expect similar results here. As in Eqs. (12.2.1 - 12.2.3), we
Problem 12.5 (continued)

Define
\[ \vec{v} = -\vec{V} \phi \]
and since
\[ \nabla \cdot \vec{v} = 0 \]
then \[ \nabla^2 \phi = 0. \]

Using our experience from the electromagnetic field problems, we guess a solution of the form
\[ \phi = \frac{A}{r} \cos \theta + Br \cos \theta \]
Then
\[ \vec{v} = \left( \frac{A}{r^2} \cos \theta - B \cos \theta \right) \vec{r} + \left( \frac{A}{r^2} \sin \theta + B \sin \theta \right) \vec{\theta} \]
Now, as \( r \to \infty \)
\[ \vec{v} = V_0 \vec{I} = V_0 (\cos \theta \vec{r} - \vec{\theta} \sin \theta) \]
Therefore
\[ B = -V_0 \]
The other boundary condition at \( r = a \) is that
\[ \nabla_r (r=a) = 0 \]
Thus
\[ A = B a^2 = -V_0 a^2 \]
Therefore
\[ \vec{v} = V_0 \cos \theta \left( 1 - \frac{a^2}{r^2} \right) \vec{r} - V_0 \sin \theta \left( 1 + \frac{a^2}{r^2} \right) \vec{\theta} \]

Part b
PROBLEM 12.5 (continued)

Part c

Using Bernoulli's law, we have

\[ \frac{1}{2} \rho v_0^2 + p_o = \frac{1}{2} \rho v^2 (1 + \frac{a^4}{r^4} - \frac{2a^2}{r^2} \cos 2\theta) + p \]

Therefore the pressure is

\[ p = p_o \left( 1 - \frac{1}{2} \frac{\rho v^2}{p_o} \left( \frac{a^4}{r^4} - \frac{2a^2}{r^2} \cos 2\theta \right) \right) \]

Part d

We choose a large rectangular surface which encloses the cylinder, but the sides of which are far away from the cylinder. We write the momentum theorem as

\[ \int \rho \vec{v} (\vec{v} \cdot \vec{n}) \, da = - \int P \, da + F \]

where \( F \) is the force which the cylinder exerts on the fluid. However, with our surface far away from the cylinder

\[ \vec{V} = \vec{V}_0 \]

and the pressure is constant

\[ p = p_o. \]

Thus, integrating over the closed surface

\[ \overline{F} = 0 \]

The force which is exerted by the fluid on the cylinder is \(-F\), which, however, is still zero.
PROBLEM 12.6

Part a

This problem is analogous to 12.5, only we are now working in spherical coordinates. As in Prob. 12.5,

\[ \mathbf{V} = - \nabla \phi \]

In spherical coordinates, we try the solution to Laplace's equation

\[ \phi = Ar \cos \theta + \frac{B}{r} \cos \theta \]

(a)

Theta is measured clockwise from the x axis.

Thus

\[ \mathbf{V} = \left( - A \cos \theta + \frac{2B}{r^2} \cos \theta \right) \hat{r} + \hat{\theta} \left( A + \frac{B}{r^3} \right) \sin \theta \]

(b)

As \( r \to \infty \)

\[ \mathbf{V} = \mathbf{V}_o \left( \hat{r} \cos \theta - \hat{\theta} \sin \theta \right) \]

(c)

Therefore \( A = - V_o \)

(d)

At \( r = a \)

\[ \mathbf{V}_r(a) = 0 \]

(e)

Thus

\[ \frac{2B}{a^3} = A = - V_o \]

or

\[ B = - \frac{V_o a^3}{2} \]

(f)

Therefore

\[ \mathbf{V} = \mathbf{V}_o \left( 1 - \frac{a^3}{r^3} \right) \cos \theta \hat{r} - \mathbf{V}_o \left( 1 + \frac{a^3}{2r^3} \right) \sin \theta \hat{\theta} \]

(g)

with

\[ r = \sqrt{x_1^2 + x_2^2 + x_3^2} \]

Part b

At \( r = a, \theta = \pi, \) and \( \phi = - \frac{\pi}{2} \)

we are given that \( p = 0 \)

At this point

\[ \mathbf{V} = 0 \]

Therefore, from Bernoulli's law

\[ p = - \frac{1}{2} \rho V_o^2 \left[ \left( 1 - \frac{a^3}{r^3} \right)^2 \cos^2 \theta + \sin^2 \theta \left( 1 + \frac{a^3}{2r^3} \right)^2 \right] \]

(h)

Part c

We realize that the pressure force acts normal to the sphere in the \(- \hat{r}\) direction.
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PROBLEM 12.6 (continued)

at \( r = a \)

\[ p = -\frac{9}{8} \rho V_o^2 \sin^2 \theta \]

We see that the magnitude of \( p \) remains unchanged if, for any value of \( \theta \), we look at the pressure at \( \theta + \pi \). Thus, by the symmetry, the force in the \( x_1 \) direction is zero,

\[ \bar{f}_1 = 0. \]

PROBLEM 12.7

\( \text{Part a} \)

We are given the potential of the velocity field as

\[ \phi = \frac{V_0}{a} x_1 x_2. \quad \bar{v} = -\nabla \phi = -\frac{V_0}{a} (x_1 \bar{1}_1 + x_2 \bar{1}_2) \]

If we sketch the equipotential lines in the \( x_1 x_2 \) plane, we know that the velocity distribution will cross these lines at right angles, in the direction of decreasing potential.

\( \text{Part b} \)

\[ \bar{a} = \frac{\overline{dv}}{\overline{dt}} = \frac{\partial \bar{v}}{\partial \bar{t}} + (\bar{v} \cdot \nabla) \bar{v} \]

\[ = \left( \frac{V_0^2}{a} \right) (x_1 \bar{1}_1 + x_2 \bar{1}_2) \]

(a)

\[ \bar{a} = \left( \frac{V_0^2}{a} \right) r \bar{1}_r \]

(b)

where

\[ r = \sqrt{x_1^2 + x_2^2} \] and \( \bar{1}_r \) is a unit vector in the radial direction.

\( \text{Part c} \)

This flow could represent a fluid impinging normally on a flat plate, located along the line

\[ x_1 + x_2 = 0. \quad \text{See sketches on next page.} \]

PROBLEM 12.8

\( \text{Part a} \)

Given that

\[ \bar{v} = \bar{1}_1 \frac{v_1}{a} + \bar{1}_2 \frac{v_2}{a} \]

(a)

we have that

\[ \bar{a} = \frac{\overline{dv}}{\overline{dt}} = \frac{\partial \bar{v}}{\partial \bar{t}} + (\bar{v} \cdot \nabla) \bar{v} \]

\[ = \left( \frac{3}{a} \frac{\partial}{\partial x_1} + v_2 \frac{3}{a} \frac{\partial}{\partial x_2} \right) \bar{v} \]

(b)
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\[ \mathbf{a} = \left( \frac{V_0^2}{a} \right) \mathbf{r} \]

----- potential
_____ velocity

Problem 12.7
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PROBLEM 12.8 (Continued)

Thus
\[
\frac{v}{a} = \frac{v}{a} \frac{x_1}{a} \frac{\tau}{a} + \left(\frac{v}{a}\right)^2 \frac{x_2}{a^2} + \frac{v}{a}\frac{x_3}{a}
\]

Part b

Using Bernoulli's law, we have
\[
p_0 = \frac{1}{2} \rho \left(\frac{v}{a}\right)^2 (x_2^2 + x_1^2) + p
\]

where
\[
p = p_0 - \frac{1}{2} \rho v^2 (x_2^2 + x_1^2)
\]

and
\[
r = \sqrt{x_2^2 + x_1^2}
\]

PROBLEM 12.9

Part a

The addition of a gravitational force will not change the velocity from that of Problem 12.8. Only the pressure will change. Therefore,
\[
\bar{v} = \bar{v} \frac{x_1}{a} x_2 + \frac{x_2}{a} x_1
\]

Part b

The boundary conditions at the walls are that the normal component of the velocity must be zero at the walls. Consider first the wall
\[x_2 - x_1 = 0\]

We take the gradient of this expression to find a normal vector to the curve. (Note that this normal vector does not have unit magnitude.)
\[
\bar{n} = \bar{I}_2 - \bar{I}_1
\]

Then
\[
\bar{v} \cdot \bar{n} = \frac{v}{a} (x_1 - x_2) = 0
\]

Thus, the boundary condition is satisfied along this wall.

Similarly, along the wall
\[x_2 + x_1 = 0\]
\[
\bar{n} = \bar{I}_2 + \bar{I}_1
\]

and
\[
\bar{v} \cdot \bar{n} = \frac{v}{a} (x_1 + x_2) = 0
\]

Thus, the boundary condition is satisfied here. Along the parabolic wall
\[
x_2^2 - x_1^2 = a^2
\]
\[
\bar{n} = x_2 \bar{I}_2 - x_1 \bar{I}_1
\]
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PROBLEM 12.9 (Continued)

\[ \frac{v \cdot n}{a} \left( x_1 x_2 - x_1 x_2 \right) = 0 \]  \hspace{1cm} (j)

Thus, we have shown that along all the walls, the fluid flows purely tangential to these walls.

PROBLEM 12.10

Part a

Along the lines \( x = 0 \) and \( y = 0 \), the normal component of the velocity must be zero. In terms of the potential, we must then have

\[ \frac{\partial \phi}{\partial x} \bigg|_{x=0} = 0 \]  \hspace{1cm} (a)

and

\[ \frac{\partial \phi}{\partial y} \bigg|_{y=0} = 0 \]  \hspace{1cm} (b)

To aid in the sketch of \( \phi(x,y) \), we realize that since at the boundary the velocity must be purely tangential, the potential lines must come in normal to the walls.

Part b

For the fluid to be irrotational and incompressible, the potential must obey
PROBLEM 12.10 (Continued)

Laplace's equation
\[ \nabla^2 \phi = 0 \quad (c) \]

From our sketch of part (a), and from the boundary conditions, we guess a solution of the form
\[ \phi = -\frac{v_0}{a} (x^2 - y^2) \quad (d) \]

where \( \frac{v_0}{a} \) is a scaling constant. By direct substitution, we see that this solution satisfies all the conditions.

Part c

For the potential of part (b), the velocity is
\[ \vec{v} = -\nabla \phi = 2 \frac{v_0}{a} (x \vec{i}_x - y \vec{i}_y) \quad (e) \]

Using Bernoulli's equation, we obtain
\[ \rho = \rho + 2 \left( \frac{v_0}{a} \right)^2 (x^2 + y^2) \quad (f) \]

The net force on the wall between \( x=c \) and \( x=d \) is
\[ \bar{f} = \int \int_{z=0}^{z=w} (\rho - \rho) \, dx \, dz \, \vec{i}_y \]

where \( w \) is the depth of the wall.

Thus
\[ \bar{f} = + \left( \frac{v_0}{a} \right)^2 \frac{w}{6} \int_c^d x^2 \, dx \, \vec{i}_y \]
\[ = + \left( \frac{v_0}{a} \right)^2 \frac{w}{6} (d^3 - c^3) \vec{i}_y \quad (g) \]

Part d

The acceleration is
\[ \vec{a} = (\nabla \times \vec{v}) \vec{v} = 2 \frac{v_0}{a} x (2 \frac{v_0}{a} \vec{i}_x) - 2 \frac{v_0}{a} y (-2 \frac{v_0}{a} y \vec{i}_y) \]

or
\[ \vec{a} = 4 \left( \frac{v_0}{a} \right)^2 (x \vec{i}_x + y \vec{i}_y) \quad (i) \]

or in cylindrical coordinates
\[ \vec{a} = 4 \left( \frac{v_0}{a} \right)^2 r \vec{i}_r \quad (j) \]
**PROBLEM 12.10 (Continued)**

Since the $\nabla \cdot \mathbf{v} = 0$, we must have

$$ V_o h = v_x(x)(h - \xi) $$

or

$$ v_x(x) = \frac{V_o h}{h - \xi} = \frac{v_o}{h} (1 + \xi) $$

**Part b**

Using Bernoulli's law, we have

$$ \frac{1}{2} \rho V_o^2 + p_o = \frac{1}{2} \rho [v_x(x)]^2 + p $$

$$ p = p_o + \frac{1}{2} \rho V_o^2 - \frac{1}{2} \rho V_o^2 (1 + \xi)^2 $$

**Part c**

We linearize $p$ around $\xi = 0$ to obtain

$$ p \approx p_o - \rho V_o^2 \frac{\xi}{h} $$

Thus

$$ T_z = -P + p_o = \rho V_o^2 \frac{\xi}{h} $$
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PROBLEM 12.11 (continued)

Thus \( T_z = C \xi \) \( \text{(g)} \)

with \( C = \frac{\rho V_o^2}{h} \)

Part d

We can write the equations of motion of the membrane as

\[
\sigma_m \frac{\partial^2 \xi}{\partial t^2} = S \frac{\partial^2 \xi}{\partial x^2} + T_z \quad \text{(h)}
\]

\[
= S \frac{\partial^2 \xi}{\partial x^2} + C \xi \quad \text{(i)}
\]

We assume

\( \xi(x,t) = \text{Re} \xi e^{j(\omega t-kx)} \) \( \text{(j)} \)

Solving for the dispersion relation, we obtain

\[
- \sigma_m \omega^2 = -Sk^2 + C
\]

or

\[
\omega = \sqrt{\left[ \frac{S}{\sigma_m} k^2 - \frac{C}{\sigma_m} \right]} \quad \text{(k)}
\]

Now, since the membrane is fixed at \( x = 0 \) and \( x = L \), we know that

\[
k = \frac{n\pi}{L} \quad n = 1, 2, 3, \ldots \quad \text{(m)}
\]

Now if

\[
S\left(\frac{\pi}{L}\right)^2 - C < 0 \quad \text{(n)}
\]

we realize that the membrane will become unstable.

So for

\[
\frac{\rho V_o^2}{h} < S\left(\frac{\pi}{L}\right)^2
\]

we have stability.

Part e

As \( \xi \) increases, the velocity of the flow above the membrane increases, since the fluid is incompressible. Through Bernoulli’s law, the pressure on the membrane must decrease, thereby increasing the net upwards force on the membrane, which tends to make \( \xi \) increase even further, thus making the membrane become unstable.
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PROBLEM 12.12

Part a

We wish to write the equation of motion for the membrane.

\[
\frac{\partial^2 \xi}{\partial t^2} = \frac{\partial^2 \xi}{\partial x^2} + p_1(\xi) - p_0 - \sigma_m g \tag{a}
\]

where

\[
T_e = \frac{\varepsilon_o}{2} \left( \frac{d}{d-\xi} \right)^2 \frac{\varepsilon_o}{2} \frac{d^2}{d^2} \left( 1 + \frac{2\xi}{d} \right)
\]

is the electric force per unit area on the membrane.

In the equilibrium \( \xi(x,t) = 0 \), we must have

\[
\cdot p_1(0) = p_0 - \frac{\varepsilon_o}{2} \left( \frac{V_0}{d} \right)^2 + \sigma_m g \tag{b}
\]

As in example 12.1.3

\[
p_1 = -\rho g \xi + C
\]

and, using the boundary condition of \( \eta \), we obtain

\[
p_1 = -\rho g \xi + \sigma_m g + p_0 - \frac{\varepsilon_o}{2} \left( \frac{V_0}{d} \right)^2 \tag{c}
\]

Part b

We are interested in calculating the perturbations in \( p_1 \) for small deflections of the membrane. From Bernoulli's law, a constant of motion of the fluid is \( D \), where

\[
D = \frac{1}{2} \rho U^2 + \sigma_m g + p_0 - \frac{\varepsilon_o}{2} \left( \frac{V_0}{d} \right)^2 \tag{d}
\]

For small perturbations \( \xi(x,t) \), the velocity in the region \( 0 < x < L \) is

\[
\nu = \frac{U d}{d+\xi}
\]

We use Bernoulli's law to write

\[
\frac{1}{2} \rho u^2 + p_1(\xi) + \rho g \xi = D \tag{e}
\]

Since we have already taken care of the equilibrium terms, we are interested only in small changes of \( p_1 \), so we omit constant terms in our linearization of \( p_1 \).

Thus

\[
p_1(\xi) = -\rho g \xi + \frac{\rho u^2 \xi}{d} \tag{f}
\]

Thus, our linearized force equation is

\[
\sigma_m \frac{\partial^2 \xi}{\partial t^2} = S \frac{\partial^2 \xi}{\partial x^2} + \left( \frac{\rho u^2}{d} - \rho g + \frac{\varepsilon_o V_0^2}{d^3} \right) \xi \tag{g}
\]

We define

\[
C = -\rho g + \frac{\rho u^2}{d} + \frac{\varepsilon_o V_0^2}{d^3}
\]

and assume solutions of the form

\[
\xi(x,t) = Re^{\frac{\hat{\xi}(x)e^{i(\omega t - kx)}}}
\]
PROBLEM 12.12 (Continued)

from which we obtain the dispersion relation

\[ \omega = \left( \frac{S}{\sigma_m} k^2 - \frac{C}{\sigma_m} \right)^{\frac{1}{2}} \]  

(h)

Since the membrane is fixed at \( x=0 \) and at \( x=L \)

\[ k = \frac{n\pi}{L}, \quad n = 1, 2, 3, \ldots \]  

(i)

If \( C < 0 \), then \( \omega \) is always real, and we can have oscillation about the equilibrium. For \( C > S \left( \frac{n}{L} \right)^2 \), then \( \omega \) will be imaginary, and the system is unstable.

Part c

The dispersion relation is thus

\[ \omega = \left( \frac{S}{\sigma_m} k^2 - \frac{C}{\sigma_m} \right)^{\frac{1}{2}} \]

Consider first \( C < 0 \)

\[ k \]

\[ \omega \]

\[ -\frac{C}{\sigma_m} \]

\[ \frac{C}{S} \]

\[ k_r \]

\[ k_i \]

\[ \omega_r \]

\[ \omega_i \]

\[ \text{complex } k \text{ for real } \omega \]

\[ \text{complex } \omega \text{ for real } k \]
PROBLEM 12.12 (Continued)

Part d

Since the membrane is not moving, one wave propagates upstream and the other propagates downstream. Thus, to find the solution we need two boundary conditions, one upstream and one downstream. If, however, both waves had propagated downstream, then causality does not allow us to apply a downstream boundary condition. This is not the case here.

PROBLEM 12.13

Part a

Since \( \nabla \cdot \mathbf{v} = 0 \), in the region \( 0 < x < L \),

\[
v_x = \frac{V_0 \frac{d}{d+\xi_1 - \xi_2}}{\xi_1 - \xi_2} \xi_1 \left[ 1 - \frac{(\xi_1 - \xi_2)}{d} \right] \tag{a}
\]

where \( d \) is the spacing between membranes. Using Bernoulli's law, we can find the pressure \( p_1 \) right below membrane 1, and pressure \( p_2 \) right above membrane 2.

Thus

\[
\frac{1}{2} \rho V_0^2 + p_0 = \frac{1}{2} \rho v_x^2 + p_1 \tag{b}
\]

and

\[
\frac{1}{2} \rho V_0^2 + p_0 = \frac{1}{2} \rho v_x^2 + p_2 \tag{c}
\]

Thus

\[
p_1 = p_2 - \frac{\rho V_0^2 (\xi_1 - \xi_2)}{d} \tag{d}
\]

We may now write the equations of motion of the membranes as

\[
\sigma_m \frac{\partial^2 \xi_1}{\partial t^2} = S \frac{\partial^2 \xi_1}{\partial x^2} + (p_1 - p_0) = S \frac{\partial^2 \xi_1}{\partial x^2} + \frac{\rho V_0^2 (\xi_1 - \xi_2)}{d} \tag{e}
\]

\[
\sigma_m \frac{\partial^2 \xi_2}{\partial t^2} = S \frac{\partial^2 \xi_2}{\partial x^2} + p_0 - p_2 = S \frac{\partial^2 \xi_2}{\partial x^2} - \frac{\rho V_0^2 (\xi_1 - \xi_2)}{d} \tag{f}
\]

Assume solutions of the form

\[
\xi_1 = Re \left\{ \hat{\xi}_1 e^{j(\omega t - kx)} \right\} \tag{g}
\]

\[
\xi_2 = Re \left\{ \hat{\xi}_2 e^{j(\omega t - kx)} \right\}
\]

Substitution of these assumed solutions into our equations of motion will yield the dispersion relation

\[
- \sigma_m \omega^2 \hat{\xi}_1 = - Sk^2 \hat{\xi}_1 + \frac{\rho V_0^2}{d} (\hat{\xi}_1 - \hat{\xi}_2) \tag{h}
\]

\[
- \sigma_m \omega^2 \hat{\xi}_2 = - Sk^2 \hat{\xi}_2 + \frac{\rho V_0^2}{d} (\hat{\xi}_2 - \hat{\xi}_1)
\]

These equations may be rewritten as
PROBLEM 12.13 (Continued)

\[
\hat{\xi}_1 \left[ -\sigma_m \omega^2 + Sk^2 - \frac{\rho V_o^2}{d} \right] + \hat{\xi}_2 \left[ + \frac{\rho V_o^2}{d} \right] = 0
\]

\[
\hat{\xi}_1 \left[ \frac{\rho V_o^2}{d} \right] + \hat{\xi}_2 \left[ -\sigma_m \omega^2 + Sk^2 - \frac{\rho V_o^2}{d} \right] = 0
\]

For non-trivial solution, the determinant of coefficients of \( \hat{\xi}_1 \) and \( \hat{\xi}_2 \) must be zero.

Thus

\[
\begin{bmatrix}
-\sigma_m \omega^2 + Sk^2 - \frac{\rho V_o^2}{d}
\end{bmatrix}^2 = \left[ \frac{\rho V_o^2}{d} \right]^2
\]

or

\[-\sigma_m \omega^2 + Sk^2 - \frac{\rho V_o^2}{d} = \pm \frac{\rho V_o^2}{d}\]

If we take the upper sign (+) on the right-hand side of the above equation, we obtain

\[
\omega = \left[ \frac{S}{\sigma_m} k^2 - \frac{2\rho V_o^2}{\sigma_m d} \right]^{1/2}
\]

We see that if \( V_o \) is large enough, \( \omega \) can be imaginary. This can happen when

\[V_o^2 > \frac{Sk^2 d}{2\rho}\]

Since the membranes are fixed at \( x=0 \) and \( x=L \)

\[k = \frac{n\pi}{L} \quad n = 1, 2, 3, \ldots \]

So the membranes first become unstable when

\[V_o^2 > \frac{S(\frac{\pi}{L})^2 d}{2\rho}\]

For this choice of sign (+), \( \hat{\xi}_1 = -\hat{\xi}_2 \), so we excite the odd mode. If we had taken the negative sign, then the even mode would be excited

\[\hat{\xi}_1 = \hat{\xi}_2\]

However, the dispersion relation is then

\[\omega = \pm \frac{S}{\sigma_m} k\]

and then we would have no instability.

Part b

The odd mode is unstable.
PROBLEM 12.14

Part a

The force equation in the y direction is
\[ \frac{\partial p}{\partial y} = -\rho g \]  \hspace{1cm} (a)

Thus
\[ p = -\rho g(y-\xi) \]  \hspace{1cm} (b)

where we have used the fact that at \( y = \xi \), the pressure is zero.

Part b

\( \nabla \cdot \mathbf{v} = 0 \) implies
\[ \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} = 0 \]  \hspace{1cm} (c)

Integrating with respect to \( y \), we obtain
\[ v_y = -\frac{\partial v_x}{\partial x} y + C \]  \hspace{1cm} (d)

where \( C \) is a constant of integration to be evaluated by the boundary condition at \( y = -a \), that
\[ v_y(y = -a) = 0 \]

since we have a rigid bottom at \( y = -a \).

Thus
\[ v_y = -\frac{\partial v_x}{\partial x} (y+a) \]  \hspace{1cm} (e)

Part c

The x-component of the force equation is
\[ \rho \frac{\partial v_x}{\partial t} = -\frac{\partial p}{\partial x} = -\rho g \frac{\partial \xi}{\partial x} \]  \hspace{1cm} (f)

or
\[ \frac{\partial v_x}{\partial t} = -g \frac{\partial \xi}{\partial x} \]  \hspace{1cm} (g)

Part d

At \( y = \xi \),
\[ v_y = \frac{\partial \xi}{\partial t} \]  \hspace{1cm} (h)

Thus, from part (b), at \( y = \xi \)
\[ \frac{\partial \xi}{\partial t} = -\frac{\partial v_x}{\partial x} (\xi+a) \]  \hspace{1cm} (i)

However, since \( \xi << a \), and \( v_x \) and \( v_y \) are small perturbation quantities, we can approximately write
\[ \frac{\partial \xi}{\partial t} = -\frac{\partial v_x}{\partial x} \]  \hspace{1cm} (j)

Part e

Our equations of motion are now
PROBLEM 12.14 (Continued)

\[ \frac{\partial \xi}{\partial t} = - a \frac{\partial v}{\partial x} \]  \hspace{1cm} (k)

\[ \frac{\partial v}{\partial t} = - g \frac{\partial \xi}{\partial x} \]  \hspace{1cm} (l)

If we take \( \partial / \partial x \) of (k) and \( \partial / \partial t \) of (l) and then simplify, we obtain

\[ \frac{\partial^2 v}{\partial t^2} = a g \frac{\partial^2 \xi}{\partial x^2} \]  \hspace{1cm} (m)

We recognize this as the wave equation for gravity waves, with phase velocity

\[ v_p = \sqrt{ag} \]  \hspace{1cm} (n)

PROBLEM 12.15

Part a

As shown in Fig. 12P.15b, the H field is in the \(- \hat{i}\) direction with magnitude:

\[ |H_s| = \frac{I_0}{2\pi r_s} \]

If we integrate the MST along the surface defined in the above figure, the only contribution will be along surface (1), so we obtain for the normal traction

\[ \tau_n = - \frac{1}{2} \mu_o |H_s|^2 = - \frac{1}{8} \frac{\mu_o I_0^2}{\pi^2 r_s^2} \]  \hspace{1cm} (b)

Part b

Since the net force on the interface must be zero, we must have

\[ \tau_n + p_{int} - p_o = 0 \]  \hspace{1cm} (c)

where \( p_{int} \) is the hydrostatic pressure on the fluid side of the interface.
Thus \( p_{\text{int}} = p_0 + \frac{1}{8} \frac{\mu I^2}{\pi^2 r^2} \) \( \text{(d)} \)

Within the fluid, the pressure \( p \) must obey the relation

\[
\frac{\partial p}{\partial z} = -\rho g \quad \text{(e)}
\]

or

\[
p = -\rho g z + C \quad \text{(f)}
\]

Let us look at the point \( z = z_0, r = R_0 \). There

\[
p = -\rho g z_0 + C = p_0 + \frac{1}{8} \frac{\mu I^2}{\pi^2 R_0^2} \quad \text{(g)}
\]

Therefore

\[
C = \rho g z_0 + p_0 + \frac{1}{8} \frac{\mu I^2}{\pi^2 R_0^2} \quad \text{(h)}
\]

Now let's look at any point on the interface with coordinates \( z_s, r_s \)

Then, by Bernoulli's law,

\[
p_0 + \frac{1}{8} \frac{\mu I^2}{\pi^2 R_0^2} + \rho g z_s = \rho g z_0 + \frac{1}{8} \frac{\mu I^2}{\pi^2 R_s^2} + p_0 + \rho g s \quad \text{(i)}
\]

Thus, the equation of the surface is

\[
\rho g z_s + \frac{1}{8} \frac{\mu I^2}{\pi^2 R_s^2} = \rho g z_0 + \frac{1}{8} \frac{\mu I^2}{\pi^2 R_0^2} \quad \text{(j)}
\]

**Part c**

The total volume of the fluid is

\[
V = \pi [R_0^2 - \left(\frac{b}{2}\right)^2]a. \quad \text{(k)}
\]

We can find the value of \( z_0 \) by finding the volume of the deformed fluid in terms of \( z_0 \), and then equating this volume to \( V \).

Thus

\[
V = \pi [R_0^2 - \left(\frac{b}{2}\right)^2]a = 2\pi \int_{r=r_0}^{r=R_0} \int_0^{z_0} r dr dz \quad \text{(l)}
\]

where

\[
r_0 = \left[\frac{1}{8} \frac{\mu I^2}{\pi^2 R_0^2} \right]^{1/2} \quad \rho g z_0 + \frac{1}{8} \frac{\mu I^2}{\pi^2 R_0^2} \quad \text{(m)}
\]

Evaluating this integral, and equating to \( V \), will determine \( z_0 \).
PROBLEM 12.16

We do an analysis similar to that of Sec. 12.2.1a, to obtain

\[ E = - \frac{I}{y} \frac{V}{w} \]  
(a)

and

\[ J = \frac{I}{y} \sigma \left( - \frac{V}{w} + vB \right) = \frac{I}{k} \frac{I}{y} \]  
(b)

Here

\[ V = IR + V_0 \]  
(c)

Thus

\[ I = \frac{vBw - V_0}{R + \frac{w}{k} \sigma} \]

The electric power out is

\[ P_e = VI = (IR + V_0)I \]

\[ = \left[ V_0 + \frac{R(vBw - V_0)}{R + \frac{w}{k} \sigma} \right] \left[ \frac{vBw - V_0}{R + \frac{w}{k} \sigma} \right] \]  
(e)

From equations (12.2.23 - 12.2.25) we have

\[ \Delta p = p(0) - p(l) = \frac{IB}{d} \]  
(f)

Thus, the mechanical power in is

\[ P_M = (\Delta p)vd = \frac{Bw(vBw - V_0)v}{R + \frac{w}{k} \sigma} \]  
(g)

Plots of \( P_E \) and \( P_M \) versus \( v \) specify the operating regions of the MHD machine.
PROBLEM 12.17

Part a

The mechanical power input is

\[ P_M = - \int \int \int \nabla p v_0 \, dx \, dy \, dz \quad (a) \]

The force equation in the steady state is

\[ - \nabla p + f^e = 0 \quad (b) \]

where

\[ f^e = - J B_0 / y_0 \quad (c) \]

Thus

\[ P_M = \int \int \int J B_0 v_0 \, dx \, dy \, dz \quad (d) \]

Now

\[ \sigma = \sigma (E_y + v_0 B_0) = \sigma (-\frac{\partial V}{\partial y} + v_0 B_0) \quad (e) \]

Integrating, we obtain

\[ P = \frac{v^2}{2} - \frac{vV}{R_L} = \frac{1}{R_L} (v_0 - V) v_0 \quad (f) \]

Part b

Defining \( \eta = \frac{P_{out}}{P_M} \), we have

\[ \eta = \frac{(V_{oc} - V) V - aV^2}{(V_{oc} - V) V_{oc}} \quad (g) \]

First, we wish to find what terminal voltage maximizes \( P_{out} \). We take

\[ \frac{\partial P_{out}}{\partial V} = 0 \] and find that

\[ V = \frac{V_{oc}}{2(1+a)} \]

maximizes \( P_{out} \).

For this value of \( V \), \( \eta \) equals

\[ \eta = \frac{1}{2} \frac{1}{1+2a} \quad (h) \]

Plotting \( \eta \) vs. \( \frac{1}{a} \) gives

![Graph showing \( \eta \) vs. \( \frac{1}{a} \)]
PROBLEM 12.17 (Continued)

Now, we wish to find what voltage will give maximum efficiency, so we take

$$\frac{2\eta}{V} = 0$$

Solving for the maximum, we obtain

$$V = V_{oc} \left[ 1 \pm \frac{a}{\sqrt{1+a}} \right]$$ \hspace{1cm} (i)

We choose the negative sign, since $V < V_{oc}$ for generator operation. We thus obtain

$$\eta = 1 + 2a - 2\sqrt{a(1+a)}$$ \hspace{1cm} (j)

Plotting $\eta$ vs. $\frac{1}{a}$, we obtain

PROBLEM 12.18

From Fig. 12P.18, we have

$$\bar{E} = \bar{V} \bar{I}$$

and

$$\bar{J} = \bar{I} \left[ \frac{\bar{V}}{\bar{w}} + \bar{v} \bar{B} \right] = \frac{\bar{I}}{\bar{L}} \bar{I} \bar{y}$$ \hspace{1cm} (b)

The $z$ component of the force equation is

$$-\frac{3p}{\bar{z}} - \frac{\bar{I}}{\bar{L}} \bar{B} = 0$$ \hspace{1cm} (c)

or

$$\Delta p = p_1 - p_o = \frac{\bar{I}B}{\bar{D}} = \Delta p_o (1 - \frac{V}{V_o})$$ \hspace{1cm} (d)

Solving for $v$, we obtain

$$v = \left(1 - \frac{\bar{I}B}{\bar{D}\Delta p_o} \right) v_o$$ \hspace{1cm} (e)
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PROBLEM 12.18 (Continued)

Thus, we have

\[
\frac{I}{LD_0} = \frac{\nu}{L_0} + B\left(1 - \frac{IB}{D_0}\right)\nu_0 \tag{f}
\]

or

\[
V = I\left(\frac{\nu}{LD_0} + \frac{B^2\nu\nu_0}{D_0}\right) - \nu_0 Bw \tag{g}
\]

Thus, for our equivalent circuit

\[
\frac{w}{v_0\omega B^2} \quad \frac{I}{D_0} \quad \frac{\nu_0}{\omega B} \tag{h}
\]

and

\[
V_{oc} = -\nu_0 Bw \tag{i}
\]

We notice that the current I in Fig. 12P.18b is not consistent with that of Fig. 12P.18a. It should be defined flowing in the other direction.

PROBLEM 12.19

Using Ampere's law

\[
H = \frac{N_o I_o + N_i I_i}{d} \tag{a}
\]

Within the fluid

\[
\bar{J} = \frac{I_i}{\frac{d}{L}} \bar{I}_z = \sigma(-\frac{V_L}{w} + \nu \frac{H}{w}) \bar{I}_z \tag{b}
\]

Simplifying, we obtain

\[
I_L \left[\frac{1}{E_d} - \frac{\sigma \nu \nu_0 N_L}{d}\right] = \frac{\sigma \nu \nu_o I_o}{d} - \frac{\sigma V}{w} \tag{c}
\]

For \(V_L\) to be independent of \(I_L\), we must have

\[
\frac{\sigma \nu \nu_o N_L}{d} = \frac{1}{E_d} \tag{d}
\]

or

\[
N_L = \frac{1}{\frac{d}{\sigma \nu \nu_0}} \tag{e}
\]

PROBLEM 12.20

We define coordinate systems as shown below.
PROBLEM 12.20 (Continued)

Now, since \( \mathbf{v} \cdot \mathbf{v} = 0 \), we have
\[
\mathbf{v}_w \cdot \mathbf{v} = \mathbf{v}_w \cdot \mathbf{v} = \mathbf{v}_w \cdot \mathbf{v} = \mathbf{v}_w \cdot \mathbf{v}
\]
In system (2),
\[
\int = \int \frac{I_2}{y_2} \frac{1}{\ell_2(d_2)} = - \sigma \left( \frac{V_2}{w_2} + v \mathbf{B} \right) \frac{1}{y_2}
\]
and
\[
\Delta P_2 = p(0_+) - p(0_-) = - \frac{I_2 B}{d_2}
\]
In system (1),
\[
\int = \int \frac{I_1}{y_1} \frac{1}{\ell_1(d_1)} = \sigma \left( \frac{V_1}{w_1} - v \mathbf{B} \right)
\]
and
\[
\Delta P_1 = p(0_+) - p(0_-) = - \frac{I_1 B}{d_1}
\]

By applying Bernoulli's law at the points \( x = 0_+ \) (right before MHD system 1) and at \( x = = \ell_1+ \) (right after MHD system 1), we obtain
\[
\frac{1}{2} \rho v_1^2 + p_1(0_-) = \frac{1}{2} \rho v_1^2 + p_1(\ell_1+)
\]
or
\[
p_1(0_-) = p_1(\ell_1+)
\]
Similarly on MHD system (2):
\[
p_2(0_-) = p_2(\ell_2+)
\]
Now,
\[
\oint \mathbf{V}_p \cdot d \ell = 0
\]

Applying this relation to a closed contour which follows the shape of the channel, we obtain
\[
\ell_1- \quad x = 0+ \quad x = \ell_1+ \quad x = 0-
\]
\[
\oint \mathbf{V}_p \cdot d \ell = \int \mathbf{V} \cdot p d \ell + \int \mathbf{V}_p \cdot d \ell + \int \mathbf{V}_p \cdot d \ell + \int \mathbf{V}_p \cdot d \ell
\]
\[
\int \mathbf{V}_p \cdot d \ell
\]
\[
\ell_2- \quad x = 0+ \quad x = \ell_2+ \quad x = 0-
\]

\[
= p_1(\ell_1- - p_1(0_+ + p_2(0_- - p_1(\ell_1+) + p_2(\ell_2+)
\]
- \( p_2(0_+) + p_1(0_- - p_2(\ell_2+)
\]

From (f) and (g) we reduce this to
\[
\Delta P_1 + \Delta P_2 = 0
\]
or
\[
\frac{1}{d_1} = \frac{1}{d_2}
\]
PROBLEM 12.20 (Continued)

Thus, we may express \( v_1 \) as

\[
\mathbf{v}_1 = \left( \frac{I_2}{w_2} + \frac{I_2}{d_2} + \frac{V_1}{w_1} \right) \frac{1}{\beta} \tag{k}
\]

We substitute this into our original equation for \( J_2 \), to obtain

\[
\frac{I_2}{w_2} = -\sigma \frac{V_2}{w_2} - \sigma \left( \frac{w_2 d_2}{w_2} \right) \left( \frac{I_2}{w_2} + \frac{V_1}{w_2} \right) \tag{l}
\]

This may be rewritten as

\[
\mathbf{v}_2 = -I_2 \sigma \left[ \frac{1}{d_2 w_2} + \frac{1}{w_2} \frac{d_2}{w_2} \right] - \frac{d_2}{w_2} \mathbf{v}_1 \tag{m}
\]

The Thevenin equivalent circuit is:

\[
\begin{array}{c}
\mathbf{I}_2 \\
\mathbf{R}_{eq} \\
\mathbf{V}_{oc} \\
\end{array}
\]

\[
\begin{array}{c}
+ \\
\mathbf{V}_2 \\
- \\
\end{array}
\]

where

\[
\mathbf{V}_{oc} = d_2 \frac{d_2}{w_1} \mathbf{V}_1
\]

and

\[
\mathbf{R}_{eq} = \frac{w_2}{d_2} \left[ \frac{1}{d_2 w_2} + \frac{w_1 d_2}{w_2} \frac{d_2}{w_1} \right]
\]

PROBLEM 12.21

For the MHD system

\[
|\mathbf{J}| = \frac{I}{LW} = \sigma \left( \frac{V}{D} - \nu \mathbf{H}_0 \right) \tag{a}
\]

and

\[
\Delta p = p_1 - p_2 = \frac{I \mu \mathbf{H}_0}{\nu} \tag{b}
\]

Now, since

\[
\int \nabla \mathbf{p} \cdot d\ell = 0 \tag{c}
\]

we must have

\[
\Delta p = k \nu = \mu \mathbf{H}_0 \mathbf{L} \sigma \left( \frac{V}{D} - \nu \mathbf{H}_0 \right) \tag{d}
\]

Solving for \( \mathbf{v} \), we obtain

\[
\mathbf{v} = \frac{\mu \mathbf{H}_0 \mathbf{L} \mathbf{V}}{D \left[ k + (\mu \mathbf{H}_0)^2 \mathbf{L} \sigma \right]} \tag{e}
\]

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PROBLEM 12.22

Part a

We assume that the fluid flows in the +x direction with velocity \( v \).

Thus

\[
\bar{J} = I_3 \bar{I} = \sigma \left( \frac{V}{d} + \nu_0 H_0 \right) \bar{i}_3
\]

(a)

where \( I \) is defined as flowing out of the positive terminal of the voltage source \( V_o \).

We write the \( x \) component of the force equation as

\[
- \frac{\partial p}{\partial x} - \frac{\nu_0 H_0}{L} \frac{\partial}{\partial x} - \rho g = 0
\]

(b)

Thus

\[
p = - \left( \frac{\nu_0 H_0}{L} + \rho g \right) x
\]

(c)

For \( \Delta p = p(0) - p(L) = 0 \)

Then

\[
\frac{\nu_0 H_0}{L} = - \rho g
\]

(d)

For the external circuit shown,

\[
V = V_o - IR + V_o
\]

(e)

Solving for \( I \) we get

\[
I = \frac{V_o}{\sigma L w} + \nu_0 H_0 = \frac{-\rho g L w}{\sigma L w + \frac{R}{d}}
\]

(f)

Solving for the velocity, \( v \), we get

\[
v = \frac{-\rho g L w}{\nu_0 H_0} \left( \frac{1}{\sigma L w + \frac{R}{d}} - \frac{V_o}{d} \right)
\]

(g)

For \( v > 0 \), then

\[
V_o < \frac{-\rho g}{\nu_0 H_0} \left( \frac{d}{\sigma} + RL \right)
\]

(h)

Part b

If the product \( V_o I > 0 \), then we are supplying electrical power to the fluid. From part (a), (f) and (h), \( V_o \) is always negative, but so is \( I \). So the product \( V_o I \) is positive.
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PROBLEM 12.23

Since the electrodes are short-circuited,
\[ \vec{J} = \vec{I}_z \frac{1}{z_d} = \sigma \vec{V}_o \vec{I}_z \]  
\[ (a) \]

In the upper reservoir
\[ p_1 = p_o + \rho g(h_1 - y) \]  
\[ (b) \]

while in the lower reservoir
\[ p_2 = p_o + \rho g(h_2 - y) \]  
\[ (c) \]

The pressure drop within the MHD system is
\[ \Delta p = p(0) - p(\ell) = \frac{IB}{d} \]  
\[ (d) \]

Integrating along the closed contour from \( y=0 \) through the duct to \( y=h \), and then back to \( y=0 \) we obtain
\[ \oint_{\Gamma} \vec{V}_p \cdot d\vec{L} = 0 = -\rho g(h_1 - h_2) + \frac{IB}{d} \ell \]  
\[ (e) \]

Thus
\[ I = \frac{\rho g(h_1 - h_2)d}{BL} \]  
\[ (f) \]

and so
\[ v = \frac{I}{\sigma IdB_o} = \frac{\rho g(h_1 - h_2)}{\sigma I^2 B_o} \]  
\[ (g) \]

PROBLEM 12.24

Part a

We define the velocity \( v_h \) as the velocity of the fluid at the top interface, where
\[ v_h = -\frac{dh}{dt} \]  
\[ (a) \]

Since \( \nabla \cdot \vec{v} = 0 \), we have
\[ \vec{V}_h \cdot \vec{A} = \vec{V}_e \cdot \vec{w} \]  
\[ (b) \]

where \( \vec{V}_e \) is the velocity of flow through the MHD generator (assumed constant). We assume that accelerations of the fluid are negligible. When we obtain the solution, we must check that these approximations are reasonable. With these approximations, the pressure in the storage tank is
\[ p = -\rho g(y-h) + p_o \]  
\[ (c) \]

where \( p_o \) is the atmospheric pressure and \( y \) the vertical coordinate. The pressure drop in the MHD generator is
\[ \Delta p = \frac{I_0 \mu H}{D} \]  
\[ (d) \]

where \( I_0 \) is defined positive flowing from right to left within the generator in the end view of Fig. 12P.24.
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PROBLEM 12.24 (continued)
We have also assumed that within the generator, \( v_e \) does not vary with position.

The current within the generator is

\[
\frac{I}{L_o} = \sigma (-\frac{IR}{w} + v_e u_o h_o)
\]

Solving for \( I \), we obtain

\[
I = \frac{v_e u_o h_o}{\frac{1}{\sigma L_o D} + \frac{R}{w}}
\]  

Now, since \( \int V_p \cdot d\ell = 0 \), we have

\[
\Delta p - \rho gh = 0
\]

Thus, using (d), (f) and (g), we obtain

\[
- \rho gh + \frac{(u_o h_o)^2}{D} \left[ \frac{1}{R \frac{1}{w} + \frac{1}{\sigma L_o D}} \right] = 0
\]  

Using (b), we finally obtain

\[
\frac{dh}{dt} + sh = 0
\]

where

\[
s = \frac{\rho g w}{(u_o h_o)^2} \frac{D}{A} \left[ RD \frac{1}{w} + \frac{1}{\sigma L_o} \right]
\]

Thus

\[
h = 10 e^{-st} , \text{ until time } \tau , \text{ when the valve closes at } h = 5.
\]

Numerically

\[
s = 7.1 \times 10^{-3} , \text{ thus } \tau \approx 100 \text{ seconds.}
\]

For our approximations to be valid, we must have

\[
\left| \frac{\partial v_h}{\partial t} \right| \ll \rho g
\]

or

\[
s^2 h \ll g.
\]

Also, we must have

\[
\left| \frac{1}{2} p v_h^2 \right| \ll \left| \rho gh \right|
\]

or

\[
\frac{1}{2} s^2 h \ll g
\]

Our other approximation was

\[
\left| \frac{\partial v_e}{\partial t} \right| < \left| \frac{I u_o h_o}{D} \right|
\]

which implies from (f) that
PROBLEM 12.24 (continued)

\[
\rho s L_o \ll \frac{(\mu_0 H_0^2)}{D \left[ \frac{R}{w} + \frac{1}{\sigma L_0 D} \right]}
\]

Substituting numerical values, we see that our approximations are all reasonable.

Part b

From (b) and (f)

\[
I = \frac{\mu_0 H_0 A}{w d} \left[ \frac{1}{\sigma L_0 D} \right] \frac{2h}{R^3 t} \exp \left( -\frac{2h}{R^3 t} \right)
\]

\[
= -650 \times 10^3 \ e^{-st} \text{ amperes.}
\]

until \( t = 100 \text{ seconds} \), where \( I = -325 \times 10^3 \text{ amperes} \). Once the valve is closed, \( I = 0 \).
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PROBLEM 12.25

Part a

Within the MHD system

\[ J = -\frac{i}{L_1 D} \frac{\dot{1}}{1} = -\sigma (\frac{V}{w} - \nu_{o} H_0) \frac{\dot{1}}{1} \]  \hspace{1cm} (a)

and

\[ \Delta p = p(0) - p(-L_1) = \frac{i \mu_{o} H_0}{D} \]  \hspace{1cm} (b)

We are considering static conditions \((v=0)\) so the pressure in tank 1 is

\[ p_1 = -\rho g (x_1 - h_1) + p_o \]  \hspace{1cm} (c)

and in tank 2 is

\[ p_2 = -\rho g (x_2 - h_2) + p_o \]  \hspace{1cm} (d)

where \(p_o\) is the atmospheric pressure,

thus

\[ i = \frac{V_o}{w \left( \frac{1}{\sigma L_1 D} + \frac{R}{w} \right)} \]  \hspace{1cm} (e)

Now since \( \oint \nabla p \cdot dl = 0 \), we must have

\[ \rho \sigma h_1 + \frac{i \mu_{o} H_0}{D} - \rho g h_2 = 0 \]  \hspace{1cm} (f)

Solving in terms of \( V_o \) we obtain

\[ V_o = \frac{\rho g (h_2 - h_1) w D}{\left( \frac{1}{\sigma L_1 D} + \frac{R}{w} \right)} \]  \hspace{1cm} (g)

For \( h_2 = .5 \) and \( h_1 = .4 \) and substituting for the given values of the parameters, we obtain

\[ V_o = 6.3 \text{ millivolts} \]

Under these static conditions, the current delivered is

\[ i = \frac{\rho g (h_2 - h_1) D}{\mu_{o} H_0} = 210 \text{ amperes} \]

and the power delivered is

\[ P_e = V_o i = \left[ \frac{\rho g (h_2 - h_1) D}{\mu_{o} H_0} \right]^2 \frac{1}{w \left( \frac{1}{\sigma L_1 D} + \frac{R}{w} \right)} = 1.33 \text{ watts} \]

Part b

We expand \( h_1 \) and \( h_2 \) around their equilibrium values \( h_{10} \) and \( h_{20} \) to obtain

\[ h_1 = h_{10} + \Delta h_1 \]

\[ h_2 = h_{20} + \Delta h_2 \]
PROBLEM 12.25 (Continued)

Since the total volume of the fluid remains constant
\[ \Delta h_2 = - \Delta h_1 \]

Since we are neglecting the acceleration in the storage tanks, we may still write
\[ p_1 = - \rho g (x_2 - h_1) + p_o \]
\[ p_2 = - \rho g (x_2 - h_2) + p_o \]

Within the MHD section, the force equation is
\[ \rho \frac{\partial \mathbf{v}}{\partial t} = -\nabla p_{\text{MHD}} + \frac{i \mu_0 M}{L_1 D} \]

Integrating with respect to \( x_1 \), we obtain
\[ \Delta p_{\text{MHD}} = p(0) - p(-L) = \frac{2 \mu_0 \mu H}{L_1 D} + \rho L_1 \frac{\partial v}{\partial t} \]

The pressure drop over the rest of the pipe is
\[ \Delta p_{\text{pipe}} = -L \rho \frac{\partial v}{\partial t} \]

Again, since \( \int \nabla \cdot \mathbf{v} \, d\mathbf{x} = 0 \), we have
\[ \rho g (h_1 - h_2) + \Delta p_{\text{MHD}} + \Delta p_{\text{pipe}} = 0 \]

For \( t > 0 \) we have
\[ \frac{2V}{w} = \frac{\mu_0 \mu H}{\sigma L_1 D + \frac{R}{w}} \]

and substituting into the above equation, we obtain
\[ \rho g (h_1 - h_2) - \rho (L_1 + L_2) \frac{3v}{\partial t} + \left( \frac{2V}{w} - \frac{\mu_0 \mu H}{\sigma L_1 D + \frac{R}{w}} \right) \frac{\mu_0 \mu H}{\sigma L_1 D + \frac{R}{w}} = 0 \]

We desire an equation just in \( \Delta h_2 \). From the \( \nabla \cdot \mathbf{v} = 0 \), we obtain
\[ \nu \frac{d \Delta h_2}{dt} = \frac{d^2 \Delta h_2}{dt^2} \]

Making these substitutions, the resultant equation of motion is
\[ \frac{d^2 \Delta h_2}{dt^2} + \frac{(\mu_0 \mu H)^2}{\rho (L_1 + L_2)D} \left[ \frac{1}{\sigma L_1 D + \frac{R}{w}} \right] \frac{d \Delta h_2}{dt} + \frac{2gw \Delta h_2}{(L_1 + L_2) \lambda} = \frac{V \mu_0 \mu H}{\rho (L_1 + L_2) \lambda} \left[ \frac{1}{\sigma L_1 D + \frac{R}{w}} \right] \]

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PROBLEM 12.25 (continued)

Solving, we obtain

\[ \Delta h_2 = \frac{V_0 H}{2p \omega D (\frac{1}{\sigma L_1 D} + \frac{R}{w})} + B_1 e^{s_1 t} + B_2 e^{s_2 t} \]  

(p)

where \( B_1 \) and \( B_2 \) are arbitrary constants to be determined by initial conditions and

\[ s_2 = -\frac{[(\mu H_o)^2]}{2p(L + L_2) D (\frac{1}{\sigma L_1 D} + \frac{R}{w})} + \sqrt{\frac{[\mu H_o]}{2p(L + L_2) D (\frac{1}{\sigma L_1 D} + \frac{R}{w})}} \]

\[ s_1 = \frac{2pgwD}{(L + L_2)A} \]  

(q)

Substituting values, we obtain approximately

\[ s_1 = -0.25 \text{ sec}^{-1} \]
\[ s_2 = -0.94 \text{ sec}^{-1} \]

The initial conditions are

\[ \Delta h_2 (t=0) = 0 \]
and

\[ \frac{d\Delta h_2}{dt} (t=0) = 0 \]

Thus, solving for \( B_1 \) and \( B_2 \) we have

\[ B_1 = \frac{-V_0 H_o}{2pgwD [\frac{1}{\sigma L_1 D} + \frac{R}{w}] (1 - \frac{s_1}{s_2})} = -0.51 \]  

(r)

\[ B_2 = \frac{-V_0 H_o}{2pgwD [\frac{1}{\sigma L_1 D} + \frac{R}{w}] (1 - \frac{s_2}{s_1})} = +1.36 \times 10^{-3} \]

Thus

\[ h_2 (t) = h_2 + \Delta h_2 (t) = 0.55 + 1.36 \times 10^{-3} e^{-025t} - 0.94 e^{-025t} \]  

(s)

From (l) we have

\[ i = \frac{2V_0 - \nu_0 H_o}{w + \frac{1}{\sigma L_1 D}} \]  

(t)

Substituting numerical values, we obtain

\[ i = 420 - 2.08 \times 10^5 (B_1 s_1 e^{s_1 t} + B_2 s_2 e^{s_2 t}) \]

\[ = 420 - 268 (e^{-025t} - e^{-0.94t}) \]  

(u)
PROBLEM 12.25 (continued)

\[ h_2(t) \]

\[ \frac{.55}{.54} \]

\[ \frac{.53}{.52} \]

\[ \frac{.51}{.50} \]

\[ t \]

\[ 1, 2, 3, 4, 5, 100, 150, 200 \]

\[ i(t) \]

\[ \frac{420}{210} \]

\[ t \]

\[ 1, 2, 3, 4, 5, 100, 150, 200, 250 \]
PROBLEM 12.25 (continued)

Our approximations were made in (h) and (k). For them to be valid, the following relations must hold:

\[ \frac{\delta^2 \Delta h}{\delta t^2} \frac{\delta}{\delta t} << 1 \]

and

\[ \int \frac{\delta v}{\delta t} + (v \cdot v) v \, d \sigma \, \frac{\delta v}{\delta t} \sqrt{A} \ll L_2 \frac{\delta v}{\delta t} \]

transition region

Substituting values, we find the first ratio to be about .001, so there our approximation is good to about .1%. In the second approximation

\[ \sqrt{A} \ll \frac{3}{2} \ll .15 \]

Here, our approximation is good only to about 15%, which provides us with an idea of the error inherent in the approximation.

PROBLEM 12.26

Part a

We use the same coordinate system as defined in Fig. 12P.25. The magnetic field through the pump is

\[ B = \frac{N_1 \mu_0}{d} \]

We integrate Newton's law across the length \( l \) to obtain

\[ \rho \frac{\delta v}{\delta t} = p(0) - p(l) + J B \]

\[ = - \frac{\Delta p_0}{v_0} v + \frac{N_1 \mu_0}{d^2} i^2 \]

Thus

\[ \frac{\delta v}{\delta t} + \frac{\Delta p_0}{\rho v_0} v = \frac{N_1 \mu_0}{d^2} i^2 \sin^2 \omega t = \frac{N_1 \mu_0}{2d^2} \sigma^2 (1 - \cos 2 \omega t) \]

Solving, we obtain

\[ v = \frac{N_1 \mu_0}{2d^2} i^2 \left[ \frac{v_0 \rho \sigma}{\Delta p_0} \frac{\Delta p_0}{\rho v_0} \cos 2 \omega t + 2 \omega \sin 2 \omega t \right] - \left( \frac{\Delta p_0}{\rho v_0} \right)^2 \frac{\Delta p_0}{\rho v_0} \sigma + 4 \omega^2 \]

Part b

The ratio \( R \) of ac to dc velocity components is:

\[ R = \frac{\Delta p_0}{v_0 \rho \sigma} \left[ \left( \frac{v_0 \rho \sigma}{\Delta p_0} \right)^2 + 4\omega^2 \right]^{1/2} \]
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PROBLEM 12.27

Part a

The magnetic field in generator (1) is upward, with magnitude
\[
B_1 = \frac{N_1 \mu_0}{a} \quad \text{and in generator (2) upward with magnitude}
\[
B_2 = \frac{N_1 \mu_0}{a} + \frac{N_2 \mu_0}{a}
\]
and in generator (2) upward with magnitude
\[
B_3 = \frac{N_1 \mu_0}{a} + \frac{N_2 \mu_0}{a}
\]
We define the voltages \(V_1\) and \(V_2\) across the terminals of the generators.
Applying Kirchoff's voltage law around the loops of wire with currents \(i_1\) and \(i_2\)
we have
\[
V_1 + N \frac{d\lambda_1}{dt} + N \frac{d\lambda_2}{dt} + i_1 R_L = 0
\]
and
\[
V_2 + N \frac{d\lambda_2}{dt} - N \frac{d\lambda_1}{dt} + i_2 R_L = 0
\]
where
\[
\lambda_1 = B_1 wb
\]
\[
\lambda_2 = B_2 wb
\]
From conservation of current we have
\[
\frac{i_1}{ab\sigma} = \frac{V_1}{wab} + VB_1
\]
and
\[
\frac{i_2}{ab\sigma} = \frac{V_2}{wab} + VB_2
\]
Combining these relations, we obtain
\[
(N^2 + N^2) \frac{wb\mu_0}{m} \frac{di_1}{dt} + i_1 \left[ \frac{w}{ab\sigma} + R_L - \frac{\nu V_o}{a} \right] + \frac{w}{a} \nu V_1 = 0
\]
and
\[
(N^2 + N^2) \frac{wb\mu_0}{m} \frac{di_2}{dt} + i_2 \left[ \frac{w}{ab\sigma} + R_L - \frac{\nu V_2}{a} \right] - \frac{w}{a} \nu V_1 = 0
\]
Part b
We combine these two first-order differential equations to obtain one second-order equation.
\[
\frac{d^2 i_2}{dt^2} + \frac{di_2}{dt} + \frac{a_1 i_2}{a} = 0
\]
where
\[
a_1 = \frac{(N^2 + N^2) \frac{wb\mu_0}{m} a}{w \nu V_2 a}
\]
PROBLEM 12.27 (continued)

\[ a_2 = 2 \left[ \frac{w}{a_0} + R_L - \frac{w_o N}{a} \right] \left[ \frac{(N^2 + N^2)b}{m V} \right] \]

\[ a_3 = \frac{N m o w}{a} \]

If we assume solutions of the form

\[ i_2 = A e^{s t} \]

Then we must have

\[ a s^2 + a s + a = 0 \quad (m) \]

or

\[ s = \frac{-a \pm \sqrt{a^2 - 4a a_3}}{2 a_1} \]

For the generators to be stable, the real part of \( s \) must be negative.
Thus

\[ a_2 > 0 \quad \text{for stability} \]

which implies the condition for stability is

\[ \frac{w}{a_0} + R_L > \frac{w_o N}{a} \quad (n) \]

Part c

When \( a_2 = 0 \)

\[ \frac{w}{a_0} + R_L = \frac{w_o N}{a} \quad (o) \]

then \( s \) is purely imaginary, so the system will operate in the sinusoidal steady state.

Then

\[ s = \pm j \sqrt{\frac{a_3}{a_1}} \]

or

\[ s = \pm j \frac{N m V}{b(N^2 + N^2)} \quad (p) \]

The length \( b \) necessary for sinusoidal operation is

\[ b = \frac{w}{a_0 \left[ \frac{w_o N}{a} - R_L \right]} \quad (q) \]

Substituting values, we obtain

\[ b = 4 \text{ meters} \]

Part d

Thus, the frequency of operation is

\[ \omega = \frac{4000}{8} = 500 \text{ rad/sec.} \]

or

\[ f = \frac{\omega}{2\pi} \approx 80 \text{ Hz.} \]
PROBLEM 12.28

Part a

The magnetic field within the generator is
\[ B = \frac{\mu_0 N i}{w} \]  
(a)

The current through the generator is
\[ J = \frac{\mathbf{I}}{\mathbf{w}} = \sigma \left( \frac{\mathbf{V}}{D} + \mathbf{V} \mathbf{B} \right) \mathbf{I} \]  
(b)

Solving for \( v \), the voltage across the channel, we obtain
\[ v = \left( \frac{D}{\sigma L} - \frac{\mathbf{V} \mathbf{N}}{w} \right) i \]  
(c)

We apply Faraday's law around the electrical circuit to obtain
\[ v + \frac{1}{C} \int i dt + i R_L = -\frac{\mu_0 N^2}{w} \mathbf{L} \frac{di}{dt} \]  
(d)

Differentiating and simplifying this equation we finally obtain
\[ \frac{d^2 i}{dt^2} + \left( \frac{R_L w}{\mu_0 N^2 \mathbf{L} \mathbf{d} C} + \frac{D}{\sigma L w} - \frac{\mu_0 N \mathbf{D} \mathbf{V}}{w} \right) \frac{di}{dt} + \frac{w}{\mu_0 N^2 \mathbf{L} \mathbf{d} C} i = 0 \]  
(e)

We assume that \( i = Re^{\hat{t} e^{st}} \).

Substituting this assumed solution back into the differential equation, we obtain
\[ s^2 + \left( \frac{R_L w}{\mu_0 N^2 \mathbf{L} \mathbf{d} C} + \frac{D}{\sigma L w} - \frac{\mu_0 N \mathbf{D} \mathbf{V}}{w} \right) s + \frac{w}{\mu_0 N^2 \mathbf{L} \mathbf{d} C} = 0 \]  
(f)

Solving, we have
\[ s = -\frac{\left( \frac{R_L w}{\mu_0 N^2 \mathbf{L} \mathbf{d} C} + \frac{D}{\sigma L w} - \frac{\mu_0 N \mathbf{D} \mathbf{V}}{w} \right)}{2} \pm \sqrt{\left( \frac{R_L w}{\mu_0 N^2 \mathbf{L} \mathbf{d} C} + \frac{D}{\sigma L w} - \frac{\mu_0 N \mathbf{D} \mathbf{V}}{w} \right)^2 - \frac{w}{\mu_0 N^2 \mathbf{L} \mathbf{d} C}} \]  
(g)

For the device to be a pure ac generator, we must have that \( s \) is purely imaginary, or
\[ R_L = \left( \frac{\mu_0 N \mathbf{D} \mathbf{V}}{w} - \frac{D}{\sigma L w} \right) \frac{\mu_0 N^2 \mathbf{L} \mathbf{d} C}{w} \]  
(h)

Part b

The frequency of operation is then
\[ \omega = \frac{w}{\mu_0 N^2 \mathbf{L} \mathbf{d} C} \]  
(i)

PROBLEM 12.29

Part a

The current within the MHD generator is
\[ \bar{J} = \frac{\mathbf{I}}{\mathbf{L} \mathbf{d} \mathbf{C}} \mathbf{y} = \sigma \left( \frac{\mathbf{V}}{w} + \mathbf{v} B \mathbf{N} \right) \mathbf{I} \]  
(a)
PROBLEM 12.29 (continued)

where $V$ is the voltage across the channel. The pressure drop along the channel is

$$\Delta p = p_i - p_o = \frac{18}{d} + \rho \frac{3v}{3t}$$

(b)

where we assume that $v$ does not vary with distance along the channel. With the switch open, we apply Faraday's law around the circuit, for which we obtain

$$V + 2iR = 0$$

(c)

Since the pressure drop is maintained constant, we solve for $v$ to obtain

$$\left(\frac{2\sigma R}{\omega} + \frac{1}{2d}\right) \frac{\partial v}{B_o} \frac{3v}{3t} + \sigma vB_o = \left(\frac{1}{2d} + \frac{2\sigma R}{\omega}\right) \frac{d}{B_o} \Delta p$$

(d)

In the steady state

$$v = \left(\frac{1}{\sigma d} + \frac{2R}{w}\right) \frac{d}{B_o} \Delta p$$

(e)

and

$$i = \frac{d}{B_o} \Delta p$$

(f)

Part b

For $t > 0$, the differential equation for $v$ is

$$\left(\frac{\sigma R}{\omega} + \frac{1}{2d}\right) \frac{\partial d}{B_o} \frac{3v}{3t} + \sigma vB_o = \left(\frac{1}{2d} + \frac{2\sigma R}{\omega}\right) \frac{d}{B_o} \Delta p$$

(g)

The general solution for $v$ is

$$v = \left(\frac{1}{\sigma d} + \frac{R}{w}\right) \frac{d}{B_o} \Delta p + A e^{-t/\tau}$$

(h)

where $\tau = \left(\frac{\sigma R}{\omega} + \frac{1}{2d}\right) \frac{2d}{\sigma B_o}$

We evaluate $A$ by realizing that at $t = 0$, the velocity must be continuous.

Therefore

$$v = \left(\frac{1}{\sigma d} + \frac{R}{w}\right) \frac{d}{B_o} \Delta p + \frac{R d}{\text{w} B_o} \Delta p e^{-t/\tau}$$

(i)

and

$$i = \Delta p \left(1 + \frac{\sigma R}{\omega} \frac{R d}{B_o} \frac{e^{-t/\tau}}{\text{w}}\right) \frac{d}{B_o}$$

(j)

PROBLEM 12.30

Part a

The magnetic field in the generator is

$$B = \frac{\mu_0 N L}{d}$$

(a)

The current within the generator is

$$\frac{i}{Ld} = \sigma \left(\frac{V}{w} + vB\right)$$

(b)
PROBLEM 12.30 (continued)

where \( V \) is the voltage across the channel. The pressure drop in the channel is

\[
\Delta p = p_1 - p_0 = \Delta p_o (1 - \frac{V}{v_o}) = \frac{IB}{d}
\]

Applying Faraday's law around the external circuit, we obtain

\[
V + i(R_L + R_C) = -\frac{d(NB\omega)}{dt} = -\frac{2\omega}{d} \mu_o N^2 \frac{di}{dt}
\]

Using (a), (b), (c) and (d), the differential equation for \( i \) is then

\[
\frac{\mu_o N^2}{d} \frac{di}{dt} + i \left[ \frac{R_L + R_C}{w} + \frac{1}{\sigma Ld} - \frac{\mu_o N^2}{d} \right] + \frac{(\mu_o N^2)^2}{d \Delta p_o} v_o i^3 = 0
\]

In the steady state, we have

\[
i^2 = \frac{[\frac{R_L + R_C}{w} + \frac{1}{\sigma Ld} - \frac{\mu_o N^2}{d}]}{\frac{(\mu_o N^2)^2}{d \Delta p_o} v_o} \Delta p_o
\]

The power dissipated in \( R_L \) is

\[
P = i^2 R_L
\]

For \( P = 1.5 \times 10^6 \), then

\[
i^2 = 0.6 \times 10^8 \text{ (amperes) }^2
\]

Substituting in values for the parameters in (f), we obtain

\[
i^2 = 0.6 \times 10^8 = \frac{(.125 + 2.5 \times 10^{-6} N^2 - 6.3 \times 10^{-6} N)40 \times 10^3}{N^2 (4 \times 10^{-8})}
\]

Rearranging (g), we obtain

\[
N^2 - 102N + 2.04 \times 10^3 = 0
\]

or

\[
N = 75, 27
\]

The most efficient solution is that one which dissipates the least power in the coil's resistance. Thus, we choose

\[
N = 27
\]

Part b

Substituting numerical values into (e), using \( N = 27 \), we obtain

\[
(1.27 \times 10^7) \frac{di}{dt} - (6 \times 10^7)i + i^3 = 0
\]

or, rewriting, we have

\[
\frac{dt}{1.27 \times 10^7} = \frac{di}{i(6 \times 10^7 - i^2)}
\]

Integrating, we obtain

\[
9.4t + C = \log \left( \frac{i^2}{6 \times 10^7 - i^2} \right)
\]

We evaluate the arbitrary constant \( C \) by realizing that at \( t=0, i = 10 \) amps
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PROBLEM 12.30 (continued)

Thus \( C = -13.3 \)

We take the anti-log of both sides of (j), and solve for \( i^2 \) to obtain

\[
i^2 = \frac{6 \times 10^7}{1 + e^{(13.3 - 9.4t)}}
\]

\( 7.75 \times 10^3 \)

\( 5.5 \times 10^3 \)

\( 10 \)

\( 1.25 \) seconds

\( t \)

Part c

For \( N = 27 \), in the steady state, we use (f) to write

\[
P = i^2 R_L = -\left[ \frac{R_L + R_C}{w} + \frac{1}{\sigma k d} - \frac{\mu_o N v_o}{d} \right] \frac{d\Delta p_o R_L}{v_o}
\]

or

\[
P = a_1 R_L - a_2 R_L^2
\]

where

\[
a_1 = -\frac{d\Delta p_o}{w} \left( \frac{R_C}{w} + \frac{1}{\sigma k d} - \frac{\mu_o N v_o}{d} \right) \approx 1.47 \times 10^9
\]

and

\[
a_2 = \frac{d\Delta p_o}{\left( \frac{\mu_o N}{d} \right)^2 v_o} \approx \frac{1}{2.85 \times 10^{-11}}
\]
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PROBLEM 12.30 (continued)

PROBLEM 12.31

Part a

With the switch open, the current through the generator is

\[ J = 0 = \frac{1}{d} \int_y \sigma (-\frac{V}{w} + v B_0) dy \]  \hspace{1cm} (a)

where \( V \) is the voltage across the channel. In the steady state, the pressure drop in the channel is

\[ \Delta p = p_1 - p_0 = \frac{1}{d} B = 0 = \Delta p_0 (1 - \frac{v}{v_0}) \]  \hspace{1cm} (b)

Thus, \( v = v_0 \) and the voltage across the channel is

\[ V = v_0 B_0 w. \]  \hspace{1cm} (c)

Part b

With the switch closed, applying Faraday's law around the circuit we obtain

\[ V = i R_L \]  \hspace{1cm} (d)

Thus

\[ \frac{i}{d} = - \frac{\sigma R_L}{w} i + \sigma v B_0 \]  \hspace{1cm} (e)

and

\[ \Delta p = \frac{1}{d} B + \rho \frac{\partial v}{\partial t} \ell = \Delta p_0 (1 - \frac{v}{v_0}) \]  \hspace{1cm} (f)

Obtaining an equation in \( v \), we have

\[ \rho \ell \frac{\partial v}{\partial t} + v \left[ \frac{\Delta p_0}{v_0} + \frac{\sigma B_0}{\ell} \right] = \Delta p_0 \]  \hspace{1cm} (g)
PROBLEM 12.31 (continued)
Solving for $v$ we obtain
\[ v = Ae^{-t/\tau} + \frac{\Delta p_o}{\left(\frac{\Delta p_o}{v_o} + \frac{w_B \omega}{R_L + R_1}\right)} \quad \text{where} \quad R_1 = \frac{w}{c_L d} \quad (h) \]
and where
\[ \tau = \frac{\Delta p_o}{v_o \left(\frac{\Delta p_o}{v_o} + \frac{w_B \omega}{R_L + R_1}\right)} \quad (i) \]
at $t = 0$, the velocity must be continuous. Therefore,
\[ A = \frac{v_0 - \frac{\Delta p_o}{\left(\frac{\Delta p_o}{v_o} + \frac{w_B \omega}{R_L + R_1}\right)}}{\frac{\Delta p_o}{v_o} + \frac{w_B \omega}{R_L + R_1}} \quad (j) \]
Now, the current is
\[ i = \frac{w_B \omega v}{R_L + R_1} \quad (k) \]
Thus
\[ i = \left( \frac{w_B \omega}{R_L + R_1} \right) \left( \frac{\Delta p_o}{v_o} + \frac{w_B \omega}{R_L + R_1} \right) \left( 1 - e^{-t/\tau} \right) + v_0 e^{-t/\tau} \quad (l) \]
PROBLEM 12.32

The current in the generator is
\[
\frac{d}{dt} \mathbf{I} = \sigma \left( \frac{\mathbf{V}}{w} - v_B \right)
\] (a)
where we assume that the \( B \) field is up and that the fluid flows counter-clockwise. We integrate Newton's law around the channel to obtain
\[
\rho \frac{\partial \mathbf{V}}{\partial t} = J B \mathbf{z}_1 = \frac{1}{d} B
\] (b)
or, using (a),
\[
\frac{\partial \mathbf{V}}{\partial t} = \frac{\mathbf{w}}{d \mathbf{z}_1} \frac{\partial I}{\partial t} + \frac{B^2 \mathbf{w}}{d \mathbf{z}_1} i
\] (c)
Integrating, we have
\[
\mathbf{V} = \frac{\mathbf{w}}{d \mathbf{z}_1} i + \frac{B^2 \mathbf{w}}{d \mathbf{z}_1} \int_0^\infty \mathbf{id}t
\] (d)
Defining \( R_1 = \frac{w}{\sigma d \mathbf{z}_1} \)
and \( C_1 = \frac{\rho d \mathbf{z}_1}{w B^2} \), we rewrite (d) as
\[
\mathbf{V} = i R_1 + \frac{1}{C_1} \int_0^\infty \mathbf{id}t
\] (e)
The equivalent circuit implied by (e) is

\[
\begin{array}{c}
R_1 \\
\hline
\mathbf{C}_1 \\
\hline
\end{array}
\]

\( \mathbf{V} \)

\( \mathbf{I} \)

PROBLEM 12.33

Part a
We assume that the capacitor is initially uncharged when the switch is closed at \( t = 0 \). The current through the capacitor is
\[
i = \frac{d}{dt} \mathbf{C} = \sigma \mathbf{d} \left( -\frac{\mathbf{V}_C}{w} + v_B \mathbf{w}_o \right)
\] (a)
or
\[
\frac{d \mathbf{V}_C}{dt} + \frac{\sigma \mathbf{d} \mathbf{v}}{w \mathbf{C}} \mathbf{V}_C = \frac{\sigma \mathbf{d} v_B \mathbf{w}_o}{C} \mathbf{w}_o
\] (b)
PROBLEM 12.33 (Continued)
The solution for \( V_C \) is
\[
V_C = v_0 B o w(1 - e^{-t/\tau}) \quad (c)
\]
with \( \tau = \frac{C}{\sigma \ell d} \), where we have used the initial condition that at \( t = 0 \), the voltage cannot change instantaneously across the capacitor. The energy stored as \( t + \infty \), is
\[
W_e = \frac{1}{2} C V_C^2 = \frac{1}{2} C(v_0 B o w)^2 \quad (d)
\]
**Part b**
The pressure drop along the fluid is
\[
\Delta p = \frac{1B_0}{d} = B_o^2 v_0 \sigma \ell de^{-t/\tau} \quad (e)
\]
The total energy supplied by the fluid source is
\[
W_f = \int_0^\infty \Delta p \ v_0 \ dw \ dt
= \int_0^\infty (v_0 B_o)^2 \ \sigma \ell w \ de^{-t/\tau} \ dt
= - \sigma \ell (v_0 B_o)^2 \ \tau w \ de^{-t/\tau} \bigg|_0^\infty \quad (f)
\]
\[
W_f = C(v_0 B_o)^2 \quad (g)
\]
**Part c**
We see that the energy supplied by the fluid source is twice that stored in the capacitor. The rest of the energy has been dissipated by the conducting fluid. This dissipated energy is
\[
W_d = \int_0^\infty V_C \ id \ dt \quad (h)
\]
\[
= \int_0^\infty + (v_0 B_o)^2 w(1 - e^{-t/\tau}) \sigma \ell d e^{-t/\tau} \ dt
= \sigma \ell w (v_0 B_o)^2 \left[ -e^{-t/\tau} + \frac{\tau}{2} e^{-2t/\tau} \right] \bigg|_0^\infty \quad (i)
\]
Therefore
\[
W_d = \frac{1}{2} C(v_0 B_o w)^2 \quad (j)
\]
Thus
\[
W_{\text{fluid}} = W_{\text{elec}} + W_{\text{dissipated}} \quad (k)
\]
As we would expect from conservation of energy.
PROBLEM 12.34

The current through the generator is

\[ i = \sigma \left( \frac{V}{w} - vB \right) \]  

(a)

Since the fluid is incompressible, and the channel has constant cross-sectional area, the velocity of the fluid does not change with position. Thus, we write Newton's law as in Eq. (12.2.41) as

\[ \rho \frac{\partial v}{\partial t} = -\nabla (p + U) + \mathbf{J} \times \mathbf{B} \]  

(b)

where \( U \) is the potential energy due to gravity. We integrate this expression along the length of the tube to obtain

\[ \rho \frac{\partial v}{\partial t} = \frac{\sigma}{d} - \rho g (x_a + x_b) \]  

(c)

Realizing that \( x_a = x_b \)

and

\[ v = \frac{dx}{dt} \]  

(d)

We finally obtain

\[ \frac{d^2 x}{dt^2} + \frac{\sigma B^2 \rho_1}{\rho_2} \frac{dx}{dt} + \frac{2g}{\kappa} x = \frac{\sigma B V}{w \rho_1} \frac{\ell_1}{\kappa} \]  

(e)

We assume the transient solution to be of the form

\[ x = \hat{x} e^{st} \]  

(f)

Substituting into the differential equation, we obtain

\[ s^2 + \frac{\sigma B^2 \rho_1}{\rho_2} s + \frac{2g}{\kappa} = 0 \]  

(g)

Solving for \( s \), we obtain

\[ s = -\frac{\sigma B^2 \rho_1}{2\rho_2} \pm \sqrt{\left(\frac{\sigma B^2 \rho_1}{2\rho_2}\right)^2 - \frac{2g}{\kappa}} \]  

(h)

Substituting the given numerical values, we obtain

\[ s_1 = -29.4 \]

\[ s_2 = -0.665 \]  

(i)

In the steady state

\[ x_a = \frac{\sigma B V \rho_1}{w \rho_2} \sim 0.075 \text{ meters} \]  

(j)

Thus the general solution is of the form

\[ x_a = 0.075 + A_1 e^{s_1 t} + A_2 e^{s_2 t} \]  

(k)

where the initial conditions to solve for \( A_1 \) and \( A_2 \) are
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PROBLEM 12.34 (continued)

\[ x_a(t=0) = 0 \]
\[ \frac{dx_a}{dt}(t=0) = 0 \]

Thus, \[ A_2 = \frac{.075 s_1}{s_2 - s_1} = -.0765 \text{ and } A_1 = -\frac{.075 s_2}{s_2 - s_1} = .00174 \]

Thus, we have:

\[ x_a = .075 + .00174e^{-29.4t} -.0765e^{-66.5t} \]

Now the current is

\[ I = \xi d\sigma \left( \frac{V}{W} - B_0 \frac{dx}{dt} \right) \]
\[ = \xi d\sigma \left[ \frac{V}{W} - B_0(s_1A_1e^{s_1t} + s_2A_2e^{s_2t}) \right] \]
\[ = 100 - 2 \times 10^3(s_1A_1e^{s_1t} + s_2A_2e^{s_2t}) \text{ amperes} \]
\[ = 100(1 + e^{-29.4t} - e^{-66.5t}) \]

Sketching, we have
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PROBLEM 12.35

The currents \( I_1 \) and \( I_2 \) are determined by the resistance of the fluid between the electrodes. Thus

\[
I_1 = \frac{V_o \sigma D x}{w} \tag{a}
\]

and

\[
I_2 = \frac{V_o \sigma D y}{w} \tag{b}
\]

The magnetic field produced by the circuit is

\[
\overline{B} = \frac{\mu_o N}{w} (I_2 - I_1) \overline{I_2} \tag{c}
\]

or

\[
\overline{B} = \frac{\mu_o N}{w} V_o \sigma D (y - x) \overline{I_2} \tag{d}
\]

From conservation of mass,

\[
y = (L - x) \tag{e}
\]

Thus

\[
\overline{B} = \frac{\mu_o N V_o \sigma D}{w^2} (L - 2x) \overline{I_2} \tag{f}
\]

The momentum equation is

\[
\rho \frac{\partial v}{\partial t} = -V(p + U) + J \times \overline{B} \tag{g}
\]

Integrating the equation along the conduit's length, we obtain

\[
\rho \frac{\partial v}{\partial t} (2L + 2a) = -\rho g (y - x) - J_v B L \tag{h}
\]

Now

\[
v = -\frac{3x}{\partial t} \tag{i}
\]

so we write:

\[
2\rho (L + a) \frac{\partial^2 x}{\partial t^2} + \left( \rho g + J_v \frac{\mu_o N V_o \sigma D}{w^2} \right) (2x - L) = 0 \tag{j}
\]

We assume solutions of the form

\[
x = Re \hat{x} e^{st} + \frac{L}{2} \tag{k}
\]

Thus

\[
s^2 + \frac{g}{(L + a)} + \frac{\mu_o N V_o \sigma D}{\rho w^2 (L + a)} J_v L = 0 \tag{l}
\]

Defining

\[
\omega_o^2 = \frac{g}{(L + a)} + \frac{\mu_o N V_o \sigma D J_v L}{\rho w^2 (L + a)} \tag{m}
\]

we have our solution in the form

\[
x = A \sin \omega_o t + B \cos \omega_o t + \frac{L}{2} \tag{n}
\]

Applying the initial conditions

\[
x(0) = L \quad \text{and} \quad \frac{dx(0)}{dt} = 0 \tag{0}
\]

we obtain

\[
x = \frac{L}{2} (1 + \cos \omega_o t) \tag{p}
\]
PROBLEM 12.36

As from Eqs. (12.2.88 - 12.2.91), we assume that

\[ \vec{v} = \vec{v}_0 \theta \]
\[ \vec{B} = B_0 \vec{z} + \vec{v}_0 \theta \]
\[ \vec{J} = \vec{J}_z + \vec{z}_j \]
\[ \vec{E} = \vec{r}_z + \vec{z}_E \]

(a)

As derived in Sec. 12.2.3, Eq. (12.2.102), we know that the equation governing Alfvén waves is

\[ \frac{\partial^2 v_\theta}{\partial t^2} = \frac{B_0^2}{\mu_0 \rho} \frac{\partial^2 v_\theta}{\partial z^2} \]

(b)

For our problem, the boundary conditions are:

at \( z = 0 \) \hspace{1cm} E_r = 0 \hspace{1cm} (c)

at \( z = \ell \) \hspace{1cm} v_\theta = \text{Re}[\Omega re^{j\omega t}] \hspace{1cm}

As in section 12.2.3, we assume

\[ v_\theta = \text{Re}[A(r) v_\theta(z)e^{j\omega t}] \]

(d)

Thus, the pertinent differential equation reduces to

\[ \frac{d^2 v_\theta}{dz^2} + k^2 v_\theta = 0 \]

where

\[ k = \frac{\Omega}{\sqrt{\frac{\mu_0 B_0^2}{r}}} \]

(e)

The solution is

\[ v_\theta = C_1 \cos k z + C_2 \sin k z \]

(f)

Imposing the boundary condition at \( z = \ell \), we obtain

\[ A(r)[C_1 \cos k \ell + C_2 \sin k \ell] = \Omega r \]

(g)

We let

\[ A(r) = \frac{r}{R} \]

(h)

and thus

\[ \Omega R = C_1 \cos k \ell + C_2 \sin k \ell \]

(i)

Now

\[ E_r = -v_\theta B_0 \]

(j)

Thus, applying the second boundary condition, we obtain

\[ v_\theta(z=0) = 0 \]

or

\[ C_1 = 0 \]

(k)

Thus

\[ C_2 = \frac{\Omega R}{\sin k \ell} \]

(\ell)

Now, using the relations

\[ E_r = -v_\theta B_0 \]

(m)
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PROBLEM 12.36 (continued)

\( E_z = 0 \)  \hspace{1cm} (n)

\( \frac{\partial E_y}{\partial z} - \frac{\partial E_z}{\partial r} = - \frac{\partial B_\theta}{\partial t} \)  \hspace{1cm} (o)

\[ \frac{1}{\mu_0} \frac{\partial B_\theta}{\partial z} = J_r \]  \hspace{1cm} (p)

\[ \frac{1}{\mu_0} \frac{\partial (rB_\theta)}{\partial r} = J_z \]  \hspace{1cm} (q)

we obtain

\( \nu_\theta = \text{Re} \left[ \frac{\Omega r}{\sin k\ell} \sin kz e^{j\omega t} \right] \)  \hspace{1cm} (r)

\( B_\theta = \text{Re} \left[ \frac{j \omega B_0 k}{\ell \sin k\ell} \cos kz e^{j\omega t} \right] \)  \hspace{1cm} (s)

\( J_r = \text{Re} \left[ \frac{j \omega B_0 k^2}{\mu_0 \ell \sin k\ell} \sin kz e^{j\omega t} \right] \)  \hspace{1cm} (t)

\( J_z = \text{Re} \left[ \frac{2 \omega B_0 k}{\mu_0 \ell \sin k\ell} \cos kz e^{j\omega t} \right] \)  \hspace{1cm} (u)

PROBLEM 12.37

Part a

We perform a similar analysis as in section 12.2.3, Eqs. (12.2.84 - 12.2.88).

From Maxwell's equation

\( \nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \)  \hspace{1cm} (a)

which yields

\[ \frac{\partial \nu_y}{\partial z} = \frac{1}{\partial t} B_x \]  \hspace{1cm} (b)

Now, since the fluid is perfectly conducting,

\( \vec{E}' = \vec{E} + \vec{v} \times \vec{B} = 0 \)  \hspace{1cm} (c)

or \( E_y = v_x B_0 \)  \hspace{1cm} (d)

Substituting, we obtain

\[ B_0 \frac{\partial \nu_x}{\partial z} = \frac{\partial B_x}{\partial t} \]  \hspace{1cm} (e)

The x component of the force equation is

\[ \rho \frac{\partial \nu_x}{\partial t} = \frac{\partial T_{xz}}{\partial z} \]  \hspace{1cm} (f)

where

\[ T_{xz} = \frac{B_0}{\mu_0} B_x \]  \hspace{1cm} (g)
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PROBLEM 12.37 (continued)

Thus
\[ \frac{\partial v_x}{\partial t} = \frac{B_o}{\mu_o} \frac{\partial B_x}{\partial z} \]  
\[ (h) \]

Eliminating \( B_x \) and solving for \( v_x \), we obtain
\[ \frac{\partial^2 v_x}{\partial t^2} = \frac{B_o^2}{\mu_o^2} \frac{\partial^2 v_x}{\partial z^2} \]
\[ (i) \]

or eliminating and solving for \( H_x \), we have
\[ \frac{\partial^2 H_x}{\partial t^2} = \frac{B_o^2}{\mu_o} \frac{\partial^2 H_x}{\partial z^2} \]
\[ (j) \]

where
\[ B_x = \mu_o H_x \]
\[ (k) \]

Part b

The boundary conditions are
\[ v_x(-L,t) = \text{Re} \, ve^{i\omega t} \]
\[ (l) \]
\[ E_y(0,t) = 0 \rightarrow v_x(0,t) = 0 \]
\[ (m) \]

We write the solution in the form
\[ v_x = A e^{i(\omega t - kx)} + B e^{i(\omega t + kx)} \]
\[ (n) \]

where
\[ k = \omega \sqrt{\frac{\mu_o \rho}{B_o^2}} \]

Applying the boundary conditions, we obtain
\[ v_x(L,t) = \text{Re} \left[ -\frac{v \sin kL}{\sin kl} \right] e^{i\omega t} \]
\[ (o) \]

Now
\[ \frac{\partial v_x}{\partial t} = -\frac{B_o}{\mu_o} \frac{\partial B_x}{\partial z} \]
\[ (p) \]

or
\[ -\frac{B_o \nu k \cos kx}{\sin kL} = i\omega \mu_o H_x \]
\[ (q) \]

Thus
\[ H_x = \text{Re} \left[ -\frac{B_o \nu k \cos kx}{\sin kL} e^{i\omega t} \right] \]
\[ (r) \]

Part c

From Maxwell's equations
\[ \nabla \times H = i_y \frac{\partial H_x}{\partial z} = \overline{J} \]
\[ (s) \]

Thus
\[ \overline{J} = i_y \text{Re} \left[ \frac{B_o \nu k^2 \sin kx}{\omega \mu_o \sin kL} e^{i\omega t} \right] \]
\[ (t) \]
PROBLEM 12.37 (continued)

Since \( \nabla \cdot \mathbf{J} = 0 \), the current must have a return path, so the walls in the x-z plane must be perfectly conducting.

Even though the fluid has no viscosity, since it is perfectly conducting, it interacts with the magnetic field such that for any motion of the fluid, currents are induced such that the magnetic force tends to restore the fluid to its original position. This shearing motion sets the neighboring fluid elements into motion, whereupon this process continues throughout the fluid.
PROBLEM 13.1

In static equilibrium, we have

\[-\nabla p - \rho g \hat{\mathbf{t}} = 0\]  \hspace{1cm} (a)

Since \( p = \rho RT \), (a) may be rewritten as

\[RT \frac{dp}{dx} + \rho g = 0\]  \hspace{1cm} (b)

Solving, we obtain

\[\rho = \rho_o e^{-\frac{g}{RT} x_1}\]  \hspace{1cm} (c)

PROBLEM 13.2

Since the pressure is a constant, Eq. (13.2.25) reduces to

\[\rho v \frac{dv}{dz} = -J_y B\]  \hspace{1cm} (a)

where we use the coordinate system defined in Fig. 13.4. Now, from Eq. (13.2.21) we obtain

\[J_y = \sigma(E_y + vB)\]  \hspace{1cm} (b)

If the loading factor \( K \), defined by Eq. (13.2.32) is constant, then

\[-K v B = + E\]  \hspace{1cm} (c)

Thus, \( J_y = \sigma v B (1-K)\)  \hspace{1cm} (d)

Then

\[\rho v \frac{dv}{dz} = -\sigma v B^2 (1-K)\]  \hspace{1cm} (e)

or

\[\rho \frac{dv}{dz} = -\sigma B^2 (1-K) = -\sigma (1-K) \frac{B_i^2 A_i}{A(z)}\]  \hspace{1cm} (f)

From conservation of mass, Eq. (13.2.24), we have

\[\rho_i v, A_i = \rho A(z) v\]  \hspace{1cm} (g)

Thus

\[\frac{\rho_i v, A_i}{v} \frac{dv}{dz} = -\sigma (1-K) B_i^2 A_i\]  \hspace{1cm} (h)

Integrating, we obtain

\[\ln v = -\sigma (1-K) B_i^2 \frac{A_i}{\rho_i v} z + C\]  \hspace{1cm} (i)

or

\[v = v_i e^{-\frac{z}{\ell_d}}\]  \hspace{1cm} (j)

where \( \ell_d = \frac{\rho_i v_i}{\sigma (1-K) B_i^2} \) and we evaluate the arbitrary constant by realizing that

\[v = v_i\] at \( z = 0 \).
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PROBLEM 13.3

Part a

We assume $T$, $B_0$, $v$, $\sigma$, $c_p$ and $c_v$ are constant. Since the electrodes are short-circuited, $E = 0$, and so

$$J_y = v B_0.$$

We use the coordinate system defined in Fig. 13P.4. Applying conservation of energy, Eq. (13.2.26), we have

$$\rho v \frac{d}{dz} \left( \frac{1}{2} v^2 \right) = 0,$$

where we have set $h = \text{constant}$. (b)

Thus, $v$ is a constant, $v = v_i$. Conservation of momentum, Eq. (13.2.25), implies

$$\frac{dp}{dz} = - \frac{v_i B_0^2}{\rho_i}.$$  

Thus, $p = - \frac{v_i B_0^2 z + p_i}{\rho_i}$

The mechanical equation of state, Eq. (13.1.10) then implies

$$p = \frac{n}{\rho} RT = - \frac{v_i B_0^2 z + p_i}{\rho_i} = \rho_i - \frac{v_i B_0^2 z}{\rho_i RT}.$$

From conservation of mass, we then obtain

$$\rho_i v_i \frac{dI_i}{dz} = \left( - \frac{v_i B_0^2 z}{\rho_i RT} + \rho_i \right) v_i \frac{dI}{dz}.$$  

Thus

$$\frac{d(z)}{\rho_i \frac{dI_i}{dz}} = \frac{\rho_i}{\rho_i - \frac{v_i B_0^2 z}{\rho_i RT}}.$$ 

Part b

Then

$$\rho(z) = \rho_i - \frac{v_i B_0^2 z}{\rho_i RT}.$$

PROBLEM 13.4

Note:

There are errors in Eqs. (13.2.16) and (13.2.31). They should read:

$$\frac{1}{M^2} \frac{d(M^2)}{dx_1} = \frac{\{(\gamma-1)(1+\gamma H^2)E_3 + \gamma [2 + (\gamma-1)H^2] v_1 B_2 \} J}{1 - M^2 \gamma \rho v_i}.$$  

and

$$\frac{1}{M^2} \frac{d(M^2)}{dx_1} = \frac{1}{(1-H^2)} \left[ \frac{\{(\gamma-1)(1+\gamma H^2)E_3 + \gamma [2 + (\gamma-1)H^2] v_1 B_2 \} J}{\gamma \rho v_i} \right. - \frac{2 + (\gamma-1)H^2}{A} \right] \frac{dA}{dx_1}.$$  

Part a

We assume that $\sigma, \gamma, B_0, K$ and $M$ are constant along the channel. Then, from the corrected form of Eq. (13.2.31), we must have

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PROBLEM 13.4 (continued)

\[
0 = \frac{1}{1 - M^2} \left\{ \left[ (\gamma - 1)(1 + \gamma M^2)(-K) + \gamma(2 + (\gamma - 1)M^2) \right] \frac{\nu B^2_o \sigma (1 - K)}{\gamma \rho p} - \frac{[2 + (\gamma - 1)M^2]}{A} \frac{dA}{dz} \right\} \quad (a)
\]

Now, using the relations

\[
\nu^2 = M^2 \gamma RT
\]

and

\[
p = \rho RT
\]

we write

\[
\frac{\nu}{\gamma \rho} = \frac{M^2}{\rho \nu}
\]

Thus, we obtain

\[
\frac{1}{A^2} \frac{dA}{dz} = \frac{\left[ (\gamma - 1)(1 + \gamma M^2)(-K) + \gamma(2 + (\gamma - 1)M^2) \right] \frac{\nu B^2_o \sigma (1 - K)}{\rho \nu A}}{2 + (\gamma - 1)M^2} \quad (c)
\]

From conservation of mass,

\[
\rho \nu A = \rho_1 \nu_1 A_1
\]

Using (d), we integrate (c) and solve for \( \frac{A(z)}{A_1} \)

to obtain

\[
\frac{A(z)}{A_1} = \frac{1}{1 - \beta_1 z}
\]

where

\[
\beta_1 = \frac{\left[ (\gamma - 1)(1 + \gamma M^2)(-K) + \gamma(2 + (\gamma - 1)M^2) \right] \frac{\nu B^2_o \sigma (1 - K)}{\rho_1 \nu_1 A_1}}{2 + (\gamma - 1)M^2}
\]

We now substitute into Eq. (13.2.27) to obtain

\[
\frac{1}{v} \frac{dv}{dz} = \frac{1}{(1 - M^2)} \left[ (\gamma - 1)(-K) + \gamma \right] \frac{\nu B^2_o \sigma (1 - K)}{\gamma \rho p} - \frac{1}{A} \frac{dA}{dz} \quad (f)
\]

Thus may be rewritten as

\[
\frac{1}{v} \frac{dv}{dz} = \frac{1}{(1 - M^2)} \left[ (\gamma - 1)(-K) + \gamma \right] \frac{\sigma B^2_o (1 - K)M^2}{\rho_1 \nu_1 A_1} - \frac{\beta_1}{A_1} \quad (g)
\]

Solving, we obtain

\[
\ln v = -\frac{\beta_2}{\beta_1} \ln(1 - \beta_1 z) + \ln v_1
\]

or

\[
\frac{v(z)}{v_1} = (1 - \beta_1 z)^{-\beta_2 / \beta_1}
\]

where

\[
\beta_2 = \frac{1}{(1 - M^2)} \frac{\left[ (\gamma - 1)(-K) + \gamma \right] \frac{\sigma B^2_o (1 - K)M^2}{\rho_1 \nu_1 A_1}}{-\beta_1}
\]

Now the temperature is related through Eq. (13.2.12), as
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PROBLEM 13.4 (continued)

\[ M^2 \gamma RT = v^2 \]  \hspace{1cm} (j)

Thus

\[ \frac{T(z)}{T_i} = \left( \frac{v}{v_i} \right)^{\frac{2}{k}} \]  \hspace{1cm} (k)

From (d), we have

\[ \frac{\rho(z)}{\rho_i} = \frac{v_i A_i}{v A} \]  \hspace{1cm} (l)

Thus, from Eq. (13.1.10)

\[ \frac{p(z)}{p_i} = \frac{v_i A_i}{v A} \frac{T}{T_i} \]  \hspace{1cm} (m)

Since the voltage across the electrodes is constant,

\[ E = - \frac{v}{w(z)} = - K v(z) B_0 \]  \hspace{1cm} (n)

or

\[ w(z) = \frac{K v(z) B_0}{v_i A_i} = \frac{v_i}{v(z)} w_1 \]  \hspace{1cm} (o)

Thus,

\[ \frac{w(z)}{w_1} = \frac{v_i}{v(z)} \]  \hspace{1cm} (p)

Then

\[ \frac{d(z)}{d_1} = \frac{A(z)}{A_i} \frac{w_i}{w(z)} \]  \hspace{1cm} (q)

Part b

We now assume that \( \sigma, \gamma, B_0, K \) and \( v \) are constant along the channel. Then, from Eq. (13.2.27) we have

\[ 0 = \frac{1}{(1-M^2)} \left\{ [(\gamma - 1)(-K) + \gamma] v_i B_0^2 \frac{(1-K)\sigma}{\gamma p} - \frac{1}{A} \frac{dA}{dz} \right\} \]  \hspace{1cm} (r)

But, from Eq. (13.2.25) we know that

\[ \frac{p}{p_i} = 1 - \frac{(1-K)\sigma v_i B_0^2 z}{p_i} = 1 - \beta_3 z \]  \hspace{1cm} (s)

where \( \beta_3 = (1-K) \frac{\sigma v_i B_0^2}{p_i} \)

Substituting the results of (b), into (a) and solving for \( \frac{A(z)}{A_i} \), we obtain

\[ \frac{A(z)}{A_i} = \frac{-p_i}{p_i} \beta_3 / \beta_4 \]  \hspace{1cm} (t)

where \( \beta_4 = [(\gamma - 1)(-K) + \gamma] \frac{v_i B_0^2}{\gamma p_i} (1-K)\sigma \)

From conservation of mass,

\[ \frac{\rho(z)}{\rho_i} = \frac{A_i}{A(z)} \]  \hspace{1cm} (u)

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PROBLEM 13.4 (continued)

and so, from Eq. (13.1.10)

\[
\frac{T(z)}{T_1} = \frac{p(z)}{p_1} \frac{\rho_1}{\rho(z)}
\]

(v)

As in (p)

\[
\frac{w(z)}{w_1} = \frac{v_1}{v(z)} = 1
\]

(w)

Thus

\[
\frac{d(z)}{d_1} = \frac{A(z)}{A_1}
\]

(x)

Part c

We wish to find the length \( \ell \) such that

\[
\frac{C_T(\ell) + \frac{1}{2} [v(\ell)]^2}{C_T(0) + \frac{1}{2} [v(0)]^2} = .9
\]

(y)

For the constant \( M \) generator of part (a), we obtain from (i) and (k)

\[
C_T(\ell) = \frac{C_T(0) - \frac{1}{2} [v(\ell)]^2}{C_T(0) + \frac{1}{2} [v(0)]^2} = .9
\]

(z)

Reducing, we obtain

\[
(1 - \beta_1 \ell) = .9
\]

(aa)

Substituting the given numerical values, we have

\( \beta_1 = .396 \) and \( \beta_2/\beta_1 = -7.3 \times 10^{-2} \)

We then solve (aa) for \( \ell \), to obtain

\( \ell \approx 1.3 \) meters

For the constant \( v \) generator of part (b), we obtain from (s), (t), (u) and (v)

\[
\frac{C_T}{p_1} \left[ \frac{p(\ell)}{p_1} \frac{\rho(\ell)}{\rho(0)} \right] + \frac{1}{2} \frac{v^2}{v_1} = .9
\]

(bb)

or

\[
\frac{C_T}{p_1} \left( 1 - \frac{\beta_4}{\beta_3} \right) + \frac{1}{2} \frac{v^2}{v_1} = .9
\]

(cc)

Substituting the given numerical values, we have
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PROBLEM 13.4 (continued)

\[ \beta_3 = .45 \quad \text{and} \quad \beta_4 / \beta_3 = .857 \]

Solving for \( \ell \), we obtain

\[ \ell \sim 1.3 \text{ meters.} \]

PROBLEM 13.5

We are given the following relations:

\[ \frac{B(z)}{B_1} = \frac{E(z)}{E_1} = \frac{w_i}{w(z)} = \frac{d_i}{d(z)} = \frac{A_i}{A(z)} \sqrt[3]{\frac{1}{2}} \]

and that \( v, \sigma, \gamma, \) and \( k \) are constant.

Part a

From Eq. (13.2.33),

\[ J = (1-K)\sigma v B \] (a)

For constant velocity, conservation of momentum yields

\[ \frac{dp}{dz} = -(1-K)\sigma v B^2 \] (b)

Conservation of energy yields

\[ \rho v C_p \frac{dT}{dz} = -K(1-K)\sigma (vB)^2 \] (c)

Using the equation of state,

\[ p = \rho R T \] (d)

we obtain

\[ \frac{dT}{dz} + \frac{\rho}{\rho_1} \frac{d\rho}{dz} = -\frac{(1-K)\sigma v B^2}{R} \] (e)

or

\[ \frac{dT}{dz} + \frac{(1-K)\sigma v B^2}{C_p} = -\frac{(1-K)\sigma v B^2}{R} \] (f)

Thus,

\[ \frac{dT}{dz} = \sigma v B^2 (1-K) \left( \frac{\frac{1}{R} + \frac{K}{C_p}}{\frac{1}{R} + \frac{K}{C_p}} \right) \] (g)

Also

\[ y^2 = \frac{B_1^2 (A_i)}{A(z)} \]

and

\[ \rho A_1 = \rho(z) A(z) \]

Therefore

\[ \frac{dT}{dz} = \frac{\sigma v B^2 (1-K)(\frac{1}{R} + \frac{K}{C_p})}{\rho_1} \rho(z) \] (h)

and

\[ \frac{\rho C_p}{\rho} \frac{dT}{dz} = -K(1-K)\sigma v \frac{B_i^2 \rho}{\rho_1} \] (i)
and so
\[
\frac{dT}{dz} = - \frac{K(1-K)\rho B_i}{\rho_1 c_p} \quad (j)
\]
Therefore
\[
T = - K(1-K)\rho B_i \rho_1 c_p z + T_1 \quad (k)
\]
Let
\[
\alpha = -K(1-K)\rho B_i \rho_1 c_p \quad (l)
\]
Then
\[
T = T_1 \left( \frac{az}{T_1} + 1 \right) \quad (m)
\]
\[
\frac{d\rho}{\rho} = \frac{\rho_B(1-K)(K - \frac{l}{R})}{\rho_1 (az + T_1)} \quad (n)
\]
We let
\[
\beta = \frac{\rho_B(1-K)(K - \frac{l}{R})}{\rho_1 c_p} \quad \frac{c_p}{KR - 1}
\]
Integrating (n), we then obtain
\[
\ln \rho = \beta \ln(az + T_1) + \text{constant}
\]
or
\[
\rho = \rho_1 \left( \frac{az}{T_1} + 1 \right) \beta \quad (o)
\]
Therefore
\[
A(z) = \frac{A_1}{\left( \frac{az}{T_1} + 1 \right) \beta} \quad (p)
\]
Part b
From (m),
\[
T(\xi) = \frac{a\xi}{T_1} = .8
\]
or
\[
\frac{a\xi}{T_1} = -.2
\]
Now
\[
\frac{\alpha}{T_1} = \frac{K(1-K)\rho B_i}{\rho_1 c_p T_1}
\]
But
\[
\frac{c_p T_1}{(1-\gamma)} = \frac{\rho_1}{\rho_1(1-\gamma)} = 2.5 \times 10^6
\]
PROBLEM 13.5 (Continued)

Thus \[ \frac{a}{T_i} = \frac{-0.5(0.5)50(700)16}{0.7(2.5 \times 10^6)} = -8.0 \times 10^{-2} \]

Solving for \( \xi \), we obtain

\[ \xi = \frac{\xi^2}{8} \times 10^2 = 1.25 \text{ meters} \]

Part c

\[ \rho = \rho_i \left( \frac{a \xi}{T_i} + 1 \right) \]

Numerically

\[ \beta = \frac{c_p}{KR} - 1 = \frac{1}{1 - \frac{1}{\gamma}} - 1 \% 6. \]

Thus

\[ \rho(z) = 0.7(1 - 0.08z)^6 \]

Then it follows:

\[ p(z) = \rho RT = \rho_i (1 - 0.08z)^7 = 5 \times 10^5(1 - 0.08z)^7 \]

\[ T(z) = T_i (1 - 0.08z) \]

From the given information, we cannot solve for \( T_i \), only for

\[ RT_i = \frac{p_i}{ho_i} = \frac{v_i^2}{\gamma M_i^2} \approx 7 \times 10^5 \]

Now

\[ M^2(z) = \frac{v_i^2}{\gamma RT(z) p(z)} = \frac{v_i^2}{\gamma p(z)} \rho(z) = \frac{v_i^2}{\gamma} \frac{\rho_i}{\rho_i} \left( \frac{a \xi}{T_i} + 1 \right)^\beta \]

\[ = \frac{0.5}{1 - 0.08z} \]

Part d

The total electric power drawn from this generator is

\[ p^e = VI = -E(z)w(z)J(z)d(z) \]

\[ = -E(z)(1-K)\sigma vB(z)d(z)w(z) \]

\[ = -E_1 w_1 (1-K)\sigma vB_1 d_1 \xi \]

But

\[ E_1 = -KvB_1 \]

Thus

\[ p^e = K(vB_1)^2 w_1 d_1 \sigma(1-K) \xi \]

\[ = 0.5(700)^216(0.5)50(0.5)1.25 \]

\[ = 61.3 \times 10^6 \text{ watts} = 61.3 \text{ megawatts} \]

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\[ \frac{T(z)}{T_1} = (1 - 0.08z) \]

\[ \rho(z) = 0.7(1 - 0.08z)^6 \]

\[ p(z) = 5 \times 10^5 (1 - 0.08z)^7 \]

\[ M^2(z) = \frac{5 \times 10^5}{(1 - 0.08z)} \]
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**PROBLEM 13.6**

**Part a**

We are given that

\[ \vec{E} = \frac{1}{x} \frac{4}{3} \frac{V_o}{L^{\frac{1}{3}}} x^{\frac{1}{3}} \]  

and

\[ \rho_e = \frac{4}{9} \frac{\varepsilon_o V_o}{L^{\frac{1}{3}}} x^{\frac{2}{3}} \]  

The force equation in the steady state is

\[ \rho_m \frac{dv}{dx} - \frac{1}{x} = \rho_e \vec{E} \]  

Since \( \rho_e/\rho_m = q/m = \) constant, we can write

\[ \frac{d}{dx}(\frac{1}{2} v^2_x) = \frac{q}{m} \frac{4}{3} \frac{V_o}{L^{\frac{1}{3}}} x^{\frac{1}{3}} \]  

Solving for \( v_x \) we obtain

\[ v_x = \sqrt{\frac{2q}{m} \frac{V_o}{(L)}} x^{\frac{1}{3}} \]  

**Part b**

The total force per unit volume acting on the accelerator system is

\[ \vec{F} = \rho_e \vec{E} \]  

Thus, the total force which the fixed support must exert is

\[ \vec{F}_{\text{total}} = - \int_{-L}^{L} FdV \]  

\[ = - \int_{-L}^{L} \frac{16}{27} \frac{\varepsilon_o V^2}{L^{\frac{8}{3}}} x^{\frac{5}{3}} \text{ Adx} \]  

\[ = - \frac{8}{9} \frac{\varepsilon_o V^2}{L^{\frac{2}{3}}} \frac{A}{x} \]  

**PROBLEM 13.7**

**Part a**

We refer to the analysis performed in section 13.2.3a. The equation of motion for the velocity is, Eq. (13.2.76),

\[ \frac{\partial^2 v}{\partial t^2} = a^2 \frac{\partial^2 v}{\partial x^2} \]  

The boundary conditions are

\( v(-L) = V_0 \cos \omega t \)
\( v(0) = 0 \)

We write the solution in the form
PROBLEM 13.7 (continued)

\[ v(x, t) = \text{Re}[A e^{i(\omega t-kx)} + B e^{i(\omega t+kx)}] \]  

where \[ k = \frac{\omega}{a} \]

Using the boundary condition at \[ x_1 = 0 \], we can alternately write the solution as

\[ v = \text{Re}[A \sin kx e^{i\omega t}] \]

Applying the other boundary condition at \[ x_1 = -L \], we finally obtain

\[ v(x_1, t) = -\frac{V_0}{\sin kL} \sin kx_1 \cos \omega t. \]

The perturbation pressure is related to the velocity through Eq. (13.2.74)

\[ \rho_0 \frac{\partial v'}{\partial t} = -\frac{\partial p'}{\partial x_1} \]

Solving, we obtain

\[ \frac{\rho_0 V_0}{\sin kL} \sin kx \sin \omega t = -\frac{\partial p'}{\partial x_1} \]

or

\[ p' = \frac{\rho_0 V_0}{k \sin kL} \cos kx \sin \omega t \]

where \( \rho_0 \) is the equilibrium density, related to the speed of sound \( a \), through Eq. (13.2.83).

Thus, the total pressure is

\[ p = p_0 + p' = p_0 + \frac{\rho_0 V_0}{k \sin kL} \cos kx \sin \omega t \]

and the perturbation pressure at \( x_1 = -L \) is

\[ p'(-L, t) = \frac{\rho_0 V_0}{\sin kL} \cos kL \sin \omega t \]

**Part b**

There will be resonances in the pressure if

\[ \sin kL = 0 \]

or

\[ kL = n\pi \quad n = 1, 2, 3, \ldots \]

Thus

\[ \omega = \frac{n\pi}{L} a \]

**PROBLEM 13.8**

**Part a**

We carry through an analysis similar to that performed in section 13.2.3b.

We assume that

\[ \bar{E} = \bar{I}_2 E_2(x_1, t) \]

\[ \bar{J} = \bar{I}_2 J_2(x_1, t) \]
PROBLEM 13.8

\[ \mathbf{B} = \frac{1}{3} [\mu_0 \mathbf{H}_0 + \mu_0 \mathbf{H}'(x_1, t)] \]

Conservation of momentum yields

\[ \frac{\rho}{\partial t} \mathbf{D} \mathbf{v} = - \frac{\partial \rho}{\partial x_1} + \mathbf{J}_2 \mathbf{\mu}_0 (\mathbf{H}_0 + \mathbf{H}') \]

Conservation of energy gives us

\[ \rho \frac{D}{Dt} (u + \frac{1}{2} v^2) = - \frac{\partial}{\partial x_1} (p v_1) + J_2 E \]

We use Ampere's and Faraday's laws to obtain

\[ \frac{\partial \mathbf{H}'}{\partial x_1} = - \mathbf{J}_2 \]

and

\[ \frac{\partial E}{\partial x_1} = - \mu_0 \frac{\partial H'}{\partial t} \]

while

Ohm's law yields

\[ J_2 = \sigma[\mathbf{E} - \mathbf{v} \mathbf{B}] \]

Since \( \sigma \to \infty \)

\[ \mathbf{E} = \mathbf{v} \mathbf{B} \]

We linearize, as in Eq. (13.2.91), so \( \mathbf{E} \approx \mathbf{v} \mathbf{\mu}_0 \mathbf{H}_0 \)

Substituting into Faraday's law

\[ \mu_0 \mathbf{H}_0 \frac{\partial v_1}{\partial x_1} = - \mu_0 \frac{\partial H'}{\partial t} \]

Linearization of the conservation of mass yields

\[ \frac{\partial \rho'}{\partial t} = - \rho_0 \frac{\partial v_1}{\partial x_1} \]

Thus, from (g)

\[ \frac{\mu_0 \mathbf{H}_0}{\rho_0} \frac{\partial \rho'}{\partial t} = \mu_0 \frac{\partial H'}{\partial t} \]

Then

\[ \frac{\mathbf{H}_0}{H_3'} = \frac{\rho_0}{\rho'} \]

Linearizing Eq. (13.2.71), we obtain

\[ \frac{D \rho'}{Dt} = \frac{\gamma \rho_0}{\rho_0} \frac{D \rho'}{Dt} \]
PROBLEM 13.8 (continued)

Defining the acoustic speed

\[ a_s = \left( \frac{yp_o}{\rho_o} \right)^{1/2} \]

where \( p_o \) is the equilibrium pressure,

\[ p_o = p_1 - \frac{\mu_{o} H_o^2}{2} \]

we have

\[ p' = a_s^2 \rho' \] (l)

Linearization of conservation of momentum (a) yields

\[ \rho_o \frac{\partial v_1}{\partial t} = -\rho_o \frac{\partial p'}{\partial x_1} - \frac{3}{2} H' \frac{\partial}{\partial x_1} \frac{\mu_{o} H_o}{\rho_o} \] (m)

or, from (j) and (l),

\[ \rho_o \frac{\partial v_1}{\partial t} = -\frac{\partial p'}{\partial x_1} \left( a_s^2 - \frac{\mu_{o} H_o^2}{\rho_o} \right) \] (n)

Differentiating (m) with respect to time, and using conservation of mass (h), we finally obtain

\[ \frac{\partial^2 v_1}{\partial t^2} = a_s^2 \frac{\partial^2 v_1}{\partial x_1^2} + \mu_{o} H_o \frac{\partial}{\partial x_1} \frac{\mu_{o} H_o^2}{\rho_o} \] (o)

Defining

\[ a^2 = a_s^2 + \frac{\mu_{o} H_o^2}{\rho_o} \] (p)

we have

\[ \frac{\partial^2 v_1}{\partial t^2} = a^2 \frac{\partial^2 v_1}{\partial x_1^2} \] (q)

Part b

We assume solutions of the form

\[ v_1 = \text{Re} \left[ A_1 e^{j(\omega t-kx_1)} + A_2 e^{j(\omega t+kx_1)} \right] \] (r)

where \( k = \frac{\omega}{a} \)

The boundary condition at \( x_1 = -L \) is

\[ v(-L,t) = v_s \cos \omega t = v_s \text{Re} e^{j\omega t} \] (s)

and at \( x_1 = 0 \)

\[ M \frac{\partial v_1}{\partial t}(0,t) = p' A \bigg|_{x_1=0} + \mu_{o} H_o H'_o A \bigg|_{x_1=0} \] (t)

From (h), (j) and (l),

\[ \frac{1}{a_s^2} \frac{\partial^2 v_1}{\partial t} = -\rho_o \frac{\partial v_1}{\partial x_1} \] (u)
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PROBLEM 13.8 (continued)

\[
\frac{H'}{H_0} = \frac{p'}{a^2 \rho_0}
\]  

Thus

\[
M \frac{dv_1(0,t)}{dt} = A \left( \frac{\mu H^2}{a^2 \rho_0} + 1 \right) p' = A \frac{a^2}{a^2} - p' 
\]  

From (u), we solve for \( p' \) to obtain:

\[
p' \bigg|_{x_1=0} = - \frac{\rho_0 a^2 k}{\omega} (A_2 - A_1) e^{j \omega t}
\]  

Substituting into (s) and (t), we have

\[
M \omega (A_1 + A_2) = A \left( \frac{\rho_0 a^2 k}{\omega} \right) (A_1 - A_2)
\]

and

\[
A_1 e^{jkx} + A_2 e^{-jkx} = v_s
\]

Solving for \( A_1 \) and \( A_2 \), we obtain

\[
A_1 = \frac{M \omega + A \rho_0}{2(\omega - \mu H \sin kl + A \rho_0 \cos kl)}
\]

and

\[
A_2 = \frac{(A \rho_0 - M \omega) v_s}{2(\omega - \mu H \sin kl + A \rho_0 \cos kl)}
\]

Thus, the velocity of the piston is

\[
v_1(0,t) = \text{Re} \left[ A_1 + A_2 e^{j \omega t} \right]
\]

\[
v_1(0,t) = \frac{A \rho_0 v_s}{\omega - \mu H \sin kl + A \rho_0 \cos kl} \cos \omega t
\]

PROBLEM 13.9

Part a

The differential equation for the velocity as derived in problem 13.8 is

\[
\frac{\partial^2 v_1}{\partial t^2} = a^2 \frac{\partial^2 v_1}{\partial x_1^2}
\]

where

\[
a^2 = a^2_s + \frac{\mu \rho_0 H^2}{\rho_0}
\]

with

\[
a^2_s = \left( \frac{\gamma p_o}{\rho_0} \right)^{1/2}
\]

where \( p_o = p_1 - \frac{\mu \rho_0 H^2}{2} \)

Part b

We assume a solution of the form
PROBLEM 13.9 (continued)

\[ V(x_1, t) = \text{Re} \left[ De^{i(\omega t - kx_1)} \right] \] where \( k = \frac{v}{a} \)

We do not consider the negatively traveling wave, as we want to use this system as a delay line without distortion. The boundary condition at \( x_1 = -L \) is

\[ V(-L, t) = \text{Re} \, V_s e^{i\omega t} \]

and at \( x_1 = 0 \) is

\[ \frac{dV(0, t)}{dt} = p'(0, t)A - Bv(0, t) + \mu H_1 H' A \quad (b) \]

From problem 13.8, (h), (j) and (l)

\[ p' = a_s^2 p' \quad , \quad \frac{\partial p'}{\partial t} = - \rho_o \frac{\partial v_1}{\partial x_1} \quad \text{and} \quad \frac{\partial H'}{\partial t} = \frac{1}{a_s^2} \frac{p'}{s_o} \]

Thus, (b) becomes

\[ - BDe^{i\omega t} + \left( \frac{a_s^2}{a} \right) p'A = 0 \quad (c) \]

where

\[ p' \bigg|_{x_1=0} = - \rho_o D(-jk) \frac{a^2}{s} \quad e^{i\omega t} \quad (d) \]

Thus, for no reflections

\[ - B + \left( \frac{a_s^2}{a} \right) \frac{A\rho_s a^2}{s_o} = 0 \quad (e) \]

or

\[ B = A\rho_s \quad (f) \]

PROBLEM 13.10

The equilibrium boundary conditions are

\[ T[-(L_1 + L_2 + \Delta), t] = T_0 \]
\[ T[-(L_1 + \Delta), t]A_s = -p_0 A_c \]

Boundary conditions for incremental motions are

1) \[ T[-(L_1 + L_2 + \Delta), t] = T_s(t) \]
2) \[ -T[-(L_1 + \Delta), t]A_s - p(-L_1, t)A_c = \frac{A}{d} v_e(-L_1, t) \]
3) \[ v_e(-L_1, t) = v_e[-(L_1 + \Delta), t] \]
   since the mass is rigid

and 4) \[ v_e(0, t) = 0 \] since the wall at \( x=0 \) is fixed.

PROBLEM 13.11

Part a

We can immediately write down the equation for perturbation velocity, using equations (13.2.76) and (13.2.77) and the results of chapters 6 and 10.
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PROBLEM 13.11 (continued)

We replace \( \partial / \partial t \) by \( \partial / \partial t + v \cdot \nabla \) to obtain

\[
\left( \frac{\partial}{\partial t} + v \cdot \nabla \right) v' = a_s^2 \frac{\partial^2 v'}{\partial x^2}
\]

Letting \( v' = Re V e^{i(\omega t-kx)} \)

we have

\[
(\omega - kV_o)^2 = a_s^2 k^2
\]

Solving for \( \omega \), we obtain

\[
\omega = k(V_o \pm a_s)
\]

Part b

Solving for \( k \), we have

\[
k = \frac{\omega}{V_o \pm a_s}
\]

For \( V_o > a_s \), both waves propagate in the positive \( x \)- direction.

PROBLEM 13.12

Part a

We assume that

\[
\begin{align*}
\vec{E} &= \overline{E_z}(x,t) \\
\vec{J} &= \overline{J_z}(x,t) \\
\vec{B} &= \overline{I_z} \left[ \frac{\partial H_0}{\partial t} + H'_0(x,t) \right]
\end{align*}
\]

We also assume that all quantities can be written in the form of Eq. (13.2.91).

\[
\rho_o \frac{\partial v_x}{\partial t} = -\frac{\partial p'}{\partial x} - \frac{\partial E_z}{\partial x} \quad \text{(conservation of momentum \( \text{linearized} \))}
\]

The relevant electromagnetic equations are

\[
\frac{\partial H'_z}{\partial x} = J_z
\]

and

\[
\frac{\partial E_z}{\partial x} = \mu_o \frac{\partial H'_z}{\partial t}
\]

and the constitutive law is

\[
J_z = \sigma(E_z + v \cdot \nabla H_0)
\]

We recognize that Eqs. (13.2.94), (13.2.96) and (13.2.97) are still valid, so

\[
\frac{1}{\rho_o} \frac{\partial \rho'}{\partial t} = -\frac{\partial v_x}{\partial x}
\]

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PROBLEM 13.12 (continued)
and
\[ p' = a_s^2 q' \quad \text{(f)} \]

Part b

We assume all perturbation quantities are of the form
\[ v_x = \Re \left[ v e^{j(\omega t - kx)} \right] \]

Using (b), (a) may be rewritten as
\[ \rho_0 \hat{v}^w = jk\hat{p} + \mu_0 \hat{H}^o jk\hat{H} \quad \text{(g)} \]

and (c) may now be written as
\[ -jk\hat{E} = \mu_0 j\omega \hat{H} \quad \text{(h)} \]

Then, from (b) and (d)
\[ -jk\hat{H} = \sigma(\hat{E} + \nu_0 \hat{H}_o) \quad \text{(i)} \]

Solving (g) and (h) for \( \hat{H} \) in terms of \( \hat{v} \), we have
\[ \hat{H} = \frac{\nu_0 \mu_0 \hat{H}_o}{-jk + \sigma \nu_0 \omega \hat{H}_o} \quad \text{(j)} \]

From (e) and (f), we solve for \( \hat{p} \) in terms of \( \hat{v} \) to be
\[ \hat{p} = \frac{k}{\omega} \rho_0 a_s^2 \hat{v} \quad \text{(k)} \]

Substituting (j) and (k) back into (g), we find
\[ \hat{v} \left( \rho_0 j\omega - \frac{jk^2}{\omega} \rho_0 a_s^2 - \frac{jk(\mu_0 H_o)^2 \sigma}{-jk + \sigma \nu_0 \omega} \right) = 0 \quad \text{(l)} \]

Thus, the dispersion relation is
\[ (\omega^2 - k^2 a_s^2) - \frac{j(\mu_0 H_o)^2 \omega k^2}{\sigma + j\mu_0 \omega} = 0 \quad \text{(m)} \]

We see that in the limit as \( \sigma \to \infty \), this dispersion relation reduces to the lossless dispersion relation
\[ \omega^2 - k^2 \left( a_s^2 + \frac{\mu_0 H_o^2}{\rho_0} \right) = 0 \quad \text{(n)} \]

Part c

If \( \sigma \) is very small, we can approximate (m) as
\[ \omega^2 - k^2 a_s^2 - \frac{j(\mu_0 H_o)^2 \omega^2}{\rho_0} \left( 1 - \frac{j\mu_0 \omega \sigma}{k^2} \right) = 0 \quad \text{(o)} \]

for which we can rewrite (o) as
PROBLEM 13.12 (continued)

\[ k^2 a_s^2 - k^2 \left[ \omega^2 - j\omega \left( \frac{\mu_o H_o}{\rho_o} \right)^2 \right] + \left( \frac{\mu_o H_o}{\rho_o} \right)^2 \omega^2 \sigma^2 \mu_o = 0 \]  

(p)

Solving for \( k^2 \), we obtain

\[ k^2 = \frac{\omega^2 - j\omega \left( \frac{\mu_o H_o}{\rho_o} \right)^2}{2 a_s^2} \pm \sqrt{ \left[ \omega^2 - j\omega \left( \frac{\mu_o H_o}{\rho_o} \right)^2 \right] \left( \frac{\mu_o H_o}{\rho_o} \right)^2 - \omega^2 \sigma^2 \mu_o } \]  

(q)

Since \( \sigma \) is very small, we expand the radical in (q) to obtain

\[ k^2 = \frac{\omega^2 - j\omega \left( \frac{\mu_o H_o}{\rho_o} \right)^2}{2 a_s^2} \pm \frac{\omega^2 - \frac{j\omega}{\rho_o} \left( \frac{\mu_o H_o}{\rho_o} \right)^2 - \frac{\omega^2}{\rho_o} \left( \frac{\mu_o H_o}{\rho_o} \right)^2}{w^2 - \frac{j\omega}{\rho_o} \left( \frac{\mu_o H_o}{\rho_o} \right)^2} \]  

(r)

Thus, our approximate solutions for \( k^2 \) are

\[ k^2 \approx \frac{\omega^2 - j\omega \left( \frac{\mu_o H_o}{\rho_o} \right)^2}{a_s^2} \]  

(s)

and

\[ k^2 \approx \frac{\left( \frac{\mu_o \omega^2 \sigma^2}{\rho_o} \right) \left( \frac{\mu_o H_o}{\rho_o} \right)^2}{w^2 - \frac{j\omega}{\rho_o} \left( \frac{\mu_o H_o}{\rho_o} \right)^2} \approx \frac{\mu_o \sigma^2}{\rho_o} \left( \frac{\mu_o H_o}{\rho_o} \right)^2 \]  

(t)

The wavenumbers for the first pair of waves are approximately:

\[ k \approx \pm \frac{\omega - j\frac{\sigma}{2\rho_o} \left( \frac{\mu_o H_o}{\rho_o} \right)^2}{a_s} \]  

(u)

while for the second pair, we obtain

\[ k \approx \pm \frac{\sigma (\mu_o H_o)}{\rho_o} \sqrt{ \frac{\mu_o}{\rho_o} } \]  

(v)

The wavenumbers from (u) represent a forward and backward traveling wave, both with amplitudes exponentially decreasing. Such waves are called 'diffusion waves'. The wavenumbers from (v) represent pure propagating waves in the forward and reverse directions.
PROBLEM 13.12 (continued)

Part d

If $\sigma$ is very large, then (m) reduces to

$$\omega^2 - k^2 a^2 - j \frac{H^2}{\rho_o} \frac{k^b}{\sigma \omega} = 0 ; \quad a^2 = a_s^2 + \frac{\mu_o H^2}{\rho_o}$$

This can be put in the form

$$k^2 = \frac{\omega^2}{a^2} - j \frac{f(\omega, k)}{k}$$

where

$$f(\omega, k) = \frac{H^2}{\rho_o \omega a^2}$$

As $\sigma$ becomes very large, the second term in (x) becomes negligible, and so

$$k^2 \approx \frac{\omega^2}{a^2}$$

However, it is this second term which represents the damping in space; that is,

$$k \approx \frac{\omega}{a} = \left[ \frac{\omega}{a} - j \frac{f(\omega, k)}{2 \sigma \omega} \right]$$

Thus, the approximate decay rate, $k_i$, is

$$k_i \approx f(\omega, k) a = \frac{H^2}{2 \sigma} \frac{a^b}{\omega}$$

or

$$k_i \approx \frac{H^2}{2 \rho_o a^2} \frac{k^b}{\omega^2} = \frac{H^2}{2 \rho_o a^2 \sigma} \omega^2$$

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PROBLEM 14.1

Part a

We can specify the relevant variables as

\[ \vec{v} = \overrightarrow{I}_1 v_1(x_2) \]
\[ \overrightarrow{E} = \overrightarrow{I}_2 E_2(x_2) + i_3 E_3(x_2) \]
\[ \overrightarrow{J} = \overrightarrow{I}_2 \overrightarrow{J}_0 \]
\[ \overrightarrow{B} = \overrightarrow{I}_2 \overrightarrow{B}_0 + \overrightarrow{I}_1 \overrightarrow{B}(x_3) \]

The \( x_1 \) component of the momentum equation is

\[ \frac{\partial^2 v_1}{\partial x_2^2} = \mu \frac{\partial v_1}{\partial x_2} \]

with solution

\[ v_1 = C_1 x_2 + C_2 \]

Applying the boundary conditions

\[ v_1 = 0 \quad @ \quad x_2 = 0 \]
\[ v_1 = v_0 \quad @ \quad x_2 = d \]

We obtain

\[ v_1 = \frac{v_0 d}{d} \]

We note that there is no magnetic force density since the imposed current and magnetic field are colinear. We apply Ohm's law for a moving fluid

\[ \overrightarrow{J} = \sigma(\overrightarrow{E} + \vec{v} \times \overrightarrow{B}) \]

in the \( x_2 \) and \( x_3 \) directions to obtain

\[ J_0 = \overrightarrow{\sigma E}_2 \]

and

\[ 0 = \sigma(E_3 + v_1 B_0) \]

since no current can flow in the \( x_3 \) direction.

Thus

\[ E_2 = \frac{J_0}{\sigma} \]

and

\[ E_3 = -\frac{v_0 d B_0}{d} \]

As from Eq. (14.2.5),

\[ V = \int E_2 dx_2 = \frac{J_0}{\sigma} d \]

Thus, the electrical input \( p_e \) per unit area in an \( x_1 - x_3 \) plane is

\[ p_e = J_0 V = \frac{J_0^2 d}{\sigma} \]
ELECTROMECHANICAL COUPLING WITH VISCOUS FLUIDS

PROBLEM 14.1 (continued)

We see that this power is dissipated as Ohmic loss. The moving fluid looks just like a resistor from the electrical terminals. The traction that must be applied to the upper plate to maintain the steady motion is

\[ \tau = \mu \frac{\partial v_1}{\partial x_2} \bigg|_{x_2 = d} = \frac{\mu v_0}{d} \]

Again we note no contribution from the magnetic forces.

The mechanical input power per unit area is then

\[ p_m = \tau v_0 = \frac{\mu v_0^2}{d} \]

The total input power per unit area is thus

\[ p_t = p_e + p_m = \frac{\mu v_0^2}{d} + \frac{J_0^2 d}{\sigma} \]

The first term is due to viscous loss that results from simple shear flow, while the second term is simply the Joule loss associated with Ohmic heating. There is no electromechanical coupling. Using the parameters from Table 14.2.1, we obtain

\[ v = 15 \text{ millivolts} \]

\[ p_t = 2.2635 \times 10^4 \text{ watts/m}^2, \text{ independent of } B_0. \]

These results correspond to the plots of Fig. 14.2.3 in the limit as \( B_0 \to 0 \).

We see that the brush losses and brush voltage are much less for this configuration than for that analyzed in Sec. 14.2.1. This is because the electrical and mechanical equations were uncoupled when the applied flux density was in the \( x_2 \) direction. This configuration is better, because low voltages at the brush eliminate arcing, and because the net power input per unit area is less no matter the field strength \( B_0 \).

The only effect of applying a flux density in the \( x_2 \) direction was to cause an electric field in the \( x_3 \) direction. However, since there was no current flow in the \( x_3 \) direction, there was no additional dissipated power. However, if \( E_3 \) became too large, the fluid might experience electrical breakdown, resulting in corona arcs.
PROBLEM 14.2

The momentum equation for the fluid is
\[ \rho \frac{\partial \vec{v}}{\partial t} + \rho (\vec{v} \cdot \nabla) \vec{v} = -\nabla p + \mu \nabla^2 \vec{v} \]  
(1)

We consider solutions of the form
\[ \vec{v} = \hat{z} v_z(r) \]
and \[ p = p(z). \]

Then in the steady state, we write the z component of (1) in cylindrical coordinates as
\[ \frac{\partial p}{\partial z} = \mu \left[ \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial v_z}{\partial r} \right) \right] \]  
(2)

Now, the left side of (2) is only a function of z, while the right side is only a function of r. Thus, from the given information
\[ \frac{\partial p}{\partial z} = \frac{p_2 - p_1}{L} \]  
(3)

Using the results of (3) in (2), we solve for \( v_z(r) \) in the form
\[ v_z(r) = \frac{p_2 - p_1}{4\mu L} \frac{1}{r} \ln r + A \]
(4)

where A and B are arbitrary constants to be evaluated by the boundary conditions
\[ v_z(r = 0) \text{ is finite} \]
and \[ v_z(r = R) = 0 \]

Thus the solution is
\[ v_z(r) = \frac{(p_2 - p_1)}{4\mu L} \left( \frac{1}{r} \ln r \right) \]  
(5)

We can also find relations between the flow rate and the pressure difference, since
\[ \int_0^R v_z 2\pi r dr = Q \]

PROBLEM 14.3

Part a

We are given the pressure drop \( \Delta p \), the magnetic field \( B_0 \), the conductivity \( \sigma \), and the dimensions of the system.

Now
\[ \int_0^d + \int_0^{-d} \int_0^d \int_0^d J_2 dx = \sigma L \int_0^d (E_1 + v_1 B_0) dx = \frac{V}{R} \]  
(6)

where \[ V = -\frac{E}{w} \]
is defined as the voltage across the resistor.

From Eq. (14.2.29), we have the solution for the velocity \( v_1 \). We then perform the integrations of (6) and solve for the voltage \( V \) to obtain
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PROBLEM 14.3(continued)

\[ V = \frac{(\Delta\rho/2d) \left( 1 - \tanh \frac{M}{M} \right)}{\frac{1}{R} + \frac{2\sigma d}{\mu \tan M}} \]  

(b)

where

\[ M = B_0 d \left( \frac{\sigma}{\mu} \right)^{1/2} \]

Then, the power \( P \) dissipated in the resistor is

\[ P = \frac{V^2}{R} = \frac{\left( \frac{\Delta\rho/2d}{B_0} \right)^2 \left( 1 - \tanh \frac{M}{M} \right)^2}{\left( \frac{1}{R} + \frac{1}{R_i} \frac{\tan M}{M} \right)^2} \]  

(c)

where we have defined the internal resistance \( R_i \) as

\[ R_i = \frac{\mu}{2\sigma d} \]

Part b

To maximize \( P \), we differentiate (c) with respect to \( R \), solve for that value of \( R \) which makes this quantity zero, and then check that this value does indeed maximize \( P \). Performing these operations, we obtain

\[ R_{\text{max}} = \frac{M R_i}{\tanh M} \]  

(d)

Part c

We must convert the given numerical values to MKS units, using the conversions

- 10,000 gauss = 1 Weber/meter
- 100 cm = 1 meter

For mercury

\[ \sigma = 10^6 \text{ mhos/m} \]

and \( \mu = 1.5 \times 10^{-3} \text{ kg/m-sec.} \)

Thus

\[ M = B_0 d \left( \frac{\sigma}{\mu} \right)^{1/2} = 2 \times 10^{-2} \left( \frac{1}{1.5 \times 10^3} \right)^{1/2} \]

\[ M = 520 \]

Then \( \tanh M \sim 1 \)

and so

\[ R_{\text{max}} = 520 \left( \frac{10^{-1}}{2 \times 10^6 \times 10^{-2}} \right) \sim 2.60 \times 10^{-3} \text{ ohms.} \]

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