In Lecture 20, we developed the Laplace transform as a generalization of the continuous-time Fourier transform. In this lecture, we introduce the corresponding generalization of the discrete-time Fourier transform. The resulting transform is referred to as the z-transform and is motivated in exactly the same way as was the Laplace transform. For example, the discrete-time Fourier transform developed out of choosing complex exponentials as basic building blocks for signals because they are eigenfunctions of discrete-time LTI systems. A more general class of eigenfunctions consists of signals of the form $z^n$, where $z$ is a general complex number. A representation of discrete-time signals with these more general exponentials leads to the z-transform.

As with the Laplace transform and the continuous-time Fourier transform, a close relationship exists between the z-transform and the discrete-time Fourier transform. For $z = e^{jn}$ or, equivalently, for the magnitude of $z$ equal to unity, the z-transform reduces to the Fourier transform. More generally, the z-transform can be viewed as the Fourier transform of an exponentially weighted sequence. Because of this exponential weighting, the z-transform may converge for a given sequence even if the Fourier transform does not. Consequently, the z-transform offers the possibility of transform analysis for a broader class of signals and systems.

As with the Laplace transform, the z-transform of a signal has associated with it both an algebraic expression and a range of values of $z$, referred to as the region of convergence (ROC), for which this expression is valid. Two very different sequences can have z-transforms with identical algebraic expressions such that their z-transforms differ only in the ROC. Consequently, the ROC is an important part of the specification of the z-transform.

Our principal interest in this and the following lectures is in signals for which the z-transform is a ratio of polynomials in $z$ or in $z^{-1}$. Transforms of this type are again conveniently described by the location of the poles (roots of the denominator polynomial) and the zeros (roots of the numerator polynomial) in the complex plane. The complex plane associated with the z-transform is referred to as the z-plane. Of particular significance in the z-plane is the circle of radius 1, concentric with the origin, referred to as the unit circle.
Since this circle corresponds to the magnitude of $z$ equal to unity, it is the contour in the $z$-plane on which the $z$-transform reduces to the Fourier transform. In contrast, for continuous time it is the imaginary axis in the $s$-plane on which the Laplace transform reduces to the Fourier transform.

The pole-zero pattern in the $z$-plane specifies the algebraic expression for the $z$-transform. In addition, the ROC must be indicated either implicitly or explicitly. There are a number of properties of the ROC in relation to the poles of the $z$-transform and in relation to characteristics of the signal in the time domain that often imply the ROC. For example, if the sequence is known to be right-sided, then the ROC must be the portion of the $z$-plane outside the circle bounded by the outermost pole. This and other properties are discussed in detail in the lecture.

**Suggested Reading**

Section 10.0, Introduction, page 629
Section 10.1, The $z$-Transform, pages 630–635
Section 10.2, The Region of Convergence for the $z$-Transform, pages 635–643
Section 10.3, The Inverse $z$-Transform, pages 643–646
The z-Transform

22.3

Discrete-Time
Fourier Transform

\[ x[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{j\omega}) e^{-j\omega n} \, d\omega \]

\[ X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n} \]

LTI Systems
Impulse response

\[ e^{j\omega n} \rightarrow h(n) e^{j\omega n} \]

\[ x[n] \rightarrow X(\omega) \]

New Notation

\[ \mathcal{X}[x[n]] = \sum_{n=0}^{\infty} x[n] e^{-j\omega n} \]

Example 10.1

\[ x[n] = a^n u[n] \]

\[ X(e^{j\omega}) = \frac{1}{1 - ae^{j\omega}} \quad |a| < 1 \]

\[ X(\omega) = \sum_{n=0}^{\infty} x[n] e^{-j\omega n} \]

Example 10.2

\[ x[n] = f\{x[n]e^{-j\omega n}\} \]

\[ X(\omega) = \mathcal{F}\{x[n]e^{-j\omega n}\} \]

\[ x[n] = \sum_{n=0}^{\infty} x[n] e^{-j\omega n} \]

\[ X(\omega) = \mathcal{F}\{x[n]e^{-j\omega n}\} \]

Example 10.3

\[ x[n] = a^n u[n] \rightarrow X(\omega) = \frac{1}{1 - ae^{-j\omega}} \quad |a| < 1 \]

Inverse z-Transform

\[ x[n] = \mathcal{F}^{-1}\{X(\omega)\} \]

\[ x[n] = \sum_{n=0}^{\infty} x[n] e^{j\omega n} \]

\[ X(\omega) = \mathcal{F}\{x[n]e^{j\omega n}\} \]

\[ x[n] = \sum_{n=0}^{\infty} x[n] e^{j\omega n} \]

\[ X(\omega) = \mathcal{F}\{x[n]e^{j\omega n}\} \]
TRANSPARENCY 22.1
The z-plane and the unit circle.

\[ z = e^{j\Omega} \]

TRANSPARENCY 22.2
The z-transform and associated pole-zero plot for a right-sided exponential sequence.
The z-Transform

22.3
The z-transform and associated pole-zero plot for a left-sided exponential sequence.

\[-a^n u[-n-1] \leftrightarrow \frac{1}{1 - az^{-1}} \quad |z| < |a|\]

Unit circle

z-plane

22.4
Pole-zero plot for a discrete-time under-damped second-order system illustrating the geometric determination of the Fourier transform from the pole-zero plot.

| H(e^{j\omega}) |
PROPERTIES OF THE REGION OF CONVERGENCE

- The ROC does not contain poles
- The ROC of $X(z)$ consists of a ring in the $z$-plane centered about the origin
- $\mathcal{F}\{x[n]\}$ converges $\iff$ ROC includes the unit circle in the $z$-plane
- $x[n]$ finite duration $\Rightarrow$ ROC is entire $z$-plane with the possible exception of $z = 0$ or $z = \infty$

- $x[n]$ right-sided and $|z| = r_o$ is in ROC $\Rightarrow$ all finite values of $z$ for which $|z| > r_o$ are in ROC
- $x[n]$ right-sided and $X(z)$ rational $\Rightarrow$ ROC is outside the outermost pole
- \( x[n] \) left-sided and \( |z| = r_o \) is in ROC
  \( \Rightarrow \) all values of \( z \) for which \( 0 < |z| < r_o \)
  will also be in ROC

- \( x[n] \) left-sided and \( X(z) \) rational
  \( \Rightarrow \) ROC inside the innermost pole

- \( x[n] \) two-sided and \( |z| = r_o \) is in ROC
  \( \Rightarrow \) ROC is a ring in the \( z \)-plane which
  includes the circle \( |z| = r_o \)

\[
X(z) = \frac{z}{(z - \frac{1}{3})(z - 2)}
\]
The ROC if the sequence is left-sided.

\[ X(z) = \frac{z}{(z - \frac{1}{3}) (z - 2)} \]

The ROC if the sequence is two-sided.

\[ X(z) = \frac{z}{(z - \frac{1}{3}) (z - 2)} \]
The $z$-Transform

22.2 (b)

\[ X(z) = \mathcal{F}\{x[n]z^{-n}\} \]

Example 10.1

\[ X[n] = a^n u[n] \]

\[ X(z) = \frac{1}{1-az^{-1}} \text{ for } |a| < 1 \]

\[ X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n} \]

\[ X(z) = \frac{1}{1-az^{-1}} \text{ for } |a| < 1 \]

\[ x[n]z^{-n} = \mathcal{F}^{-1}\{X(z)\} \]

Inverse $z$-Transform

\[ x[n] = \frac{1}{2\pi i} \oint_{|z|=\rho} X(z)z^{n-1}dz \]

Example 10.2

\[ X(z) = \mathcal{F}\{x[n]z^{-n}\} \]

\[ x[n] = \sum_{n=0}^{\infty} x[n]z^{-n} \]

ROC

\[ \sum_{n=0}^{\infty} x[n]z^{-n} = \frac{1}{1-az^{-1}} \text{ for } |a| > 1 \]

\[ x[n] = \frac{1}{2\pi i} \oint_{|z|=\rho} X(z)z^{n-1}dz \]