THE DISCRETE FOURIER TRANSFORM

Solution 9.1

(a)
\[ X(k) = \sum_{n=0}^{N-1} x(n) W_N^{kn} R_N(k) \]
\[ = R_N(k) \]

(b)
\[ X(k) = W_N^{kn_0} R_N(k) \]

(c)
\[ X(k) = \sum_{n=0}^{N-1} a^n W_N^{kn} R_N(k) \]
\[ = \left[ \frac{1-a^{-N} W_N^{kN}}{1-a W_N^{k}} \right] R_N(k) \]
\[ = \left[ \frac{1-a^{-N} W_N^{kN}}{1-a W_N^{k}} \right] R_N(k) \]

Solution 9.2

Figure S9.2-1
Solution 9.3

Since \( X_I(k) = -X_I((N - k))_N R_N(k) \)

\( X_I(0) = -X_I((N))_N R_N(k) = -X_I(0) \)

Therefore \( X_I(0) = 0 \)

Also, for \( N \) even,

\[ X_{I\frac{N}{2}} = -X_I((N - N\frac{N}{2}))_N R_N(k) = -X_{I\frac{N}{2}} \]

Therefore \( X_{I\frac{N}{2}} = 0 \).

Solution 9.4

Figure S9.4-1

Solution 9.5

Figure S9.5-1
Note that this corresponds to $x_1(n)$ circularly shifted to the right by two points.

Solution 9.6

We wish to compute $X_1(k)$ given by

$$X_1(k) = \sum_{n=0}^{9} x(n) Z_k^{-n} R_{10}(k)$$

where $Z_k = 0.5 e^{j \frac{\pi}{10} e^{j10}}$

so that

$$X_1(k) = \sum_{n=0}^{9} x(n) \left[ \frac{1}{2} e^{j \frac{2 \pi k}{10}} e^{j \frac{\pi}{10}} \right]^{-n} R_{10}(k)$$

Thus $X_1(k)$ is the 10-point DFT of the sequence

$$x_1(n) = x(n) \left[ \frac{1}{2} e^{j \frac{\pi}{10}} \right]^{-n}$$

Solution 9.7

In all of the following equations the DFT computed is valid only in the range $0 \leq k < N-1$ and is zero outside that range. This permits us to keep the equations somewhat cleaner by suppressing the use of the function $R_N(k)$.

$$G_1(k) = \sum_{n=0}^{N-1} x(N-1-n) W_N^{kn}$$

$$= \sum_{m=0}^{N-1} x(m) W_N^{k(N-1-m)}$$

$$= \sum_{m=0}^{N-1} x(m) \left( \frac{j 2 \pi k}{N} e^{j 2 \pi m k} \right)$$

$$= e^{j \frac{2 \pi k}{N}} X(e^{-j \frac{\pi k}{N}}) = H_7(k)$$

$$G_2(k) = \sum_{n=0}^{N-1} (-1)^n x(n) W_N^{kn}$$

$$= \sum_{n=0}^{N-1} x(n) W_N^{N n k}$$

$$= \sum_{n=0}^{N-1} x(n) W_N^{N n k}$$

s9.3
\[ \begin{align*}
&= \sum_{n=0}^{N-1} x(n) e^{-j\frac{2\pi}{N} (k+\frac{N}{2})n} = X(e^{j\frac{2\pi}{N} (k+\frac{N}{2})}) \\
&= H_8(k) \\
G_3(k) &= \sum_{n=0}^{N-1} x(n) W_{2N}^{nk} + \sum_{n=N}^{2N-1} x(n - N) W_{2N}^{nk} \\
&= \sum_{n=0}^{N-1} x(n) \left[ W_{2N}^{nk} + W_{2N}^{(n+N)k} \right] \\
&= \left[ 1 + (-1)^n \right] X(e^{j\frac{2N\pi}{2N} k}) = H_3(k) \\
G_4(k) &= \sum_{n=0}^{\frac{N-1}{2}} \left( x(n) + x(n + \frac{N}{2}) \right) W_{N/2}^{nk} \\
&= \sum_{n=0}^{\frac{N-1}{2}} x(n) W_{N/2}^{nk} + \sum_{n=\frac{N}{2}}^{N-1} x(n) W_{N/2}^{(n-N/2)k} \\
&= \sum_{n=0}^{N-1} x(n) W_{N/2}^{nk} = X(e^{j\frac{4\pi}{N} k}) = H_6(k) \\
G_5(k) &= \sum_{n=0}^{2N-1} x(n) W_{2N}^{nk} = X(e^{j\frac{\pi}{N} k}) = H_2(k) \\
G_6(k) &= \sum_{n=0}^{N-1} x(n) W_{2N}^{2nk} = X(e^{j\frac{2\pi}{N} k}) = H_1(k) \\
G_7(k) &= \sum_{n=0}^{N-1} x(2n) W_{N/2}^{nk} \\
&= \sum_{n=0}^{N-1} x(n) \left[ \frac{1+(-1)^n}{2} \right] W_{N/2}^{nk/2} \\
&= \frac{1}{2} \sum_{n=0}^{N-1} x(n) \left[ W_{N/2}^{nk} + W_{N/2}^{n(k+N/2)} \right]
\end{align*} \]
\[
\frac{1}{2} \left[ X \left( e^{j \frac{2\pi k}{N}} \right) + X \left( e^{j \frac{2\pi (k+N/2)}{N}} \right) \right] = H_5(k)
\]

All of the above properties can alternatively be obtained from the basic DFT properties of sections 8.7 and 8.8, or the z-transform properties of section 4.4. Many of the properties used in this problem have important practical applications. \( g_5(n) \), for example, corresponds to augmenting a finite length sequence with zeros so that a computation of the DFT for this augmented sequence provides finer spectral sampling of the Fourier transform.