LECTURE 2: Conditioning and Bayes’ rule

- Conditional probability

- Three important tools:
  - Multiplication rule
  - Total probability theorem
  - Bayes’ rule (→ inference)
The idea of conditioning

Use new information to revise a model

Assume 12 equally likely outcomes

If told $B$ occurred:

$$P(A) = \frac{5}{12} \quad P(B) = \frac{6}{12}$$

$$P(A \mid B) = \quad P(B \mid B) =$$
Definition of conditional probability

- \( P(A \mid B) = \) “probability of \( A \), given that \( B \) occurred”

\[
P(A \mid B) = \frac{P(A \cap B)}{P(B)}
\]

defined only when \( P(B) > 0 \)
Example: two rolls of a 4-sided die

- Let $B$ be the event: $\min(X, Y) = 2$

  Let $M = \max(X, Y)$

  $P(M = 1 \mid B) = \boxed{\text{}}$

  $P(M = 3 \mid B) = \boxed{\text{}}$
Conditional probabilities share properties of ordinary probabilities

\[ P(A \mid B) \geq 0 \quad \text{assuming } P(B) > 0 \]

\[ P(\Omega \mid B) = \]

\[ P(B \mid B) = \]

If \( A \cap C = \emptyset \), then \( P(A \cup C \mid B) = P(A \mid B) + P(C \mid B) \)
Models based on conditional probabilities

Event $A$: Airplane is flying above
Event $B$: Something registers on radar screen

- $P(A \cap B) =$
- $P(B) =$
- $P(A | B) =$
The multiplication rule

\[ P(A | B) = \frac{P(A \cap B)}{P(B)} \]

\[ P(A \cap B) = P(B) P(A | B) \]

\[ = P(A) P(B | A) \]

\[ P(A^c \cap B \cap C^c) = \]
Total probability theorem

- Partition of sample space into $A_1, A_2, A_3$
- Have $P(A_i)$, for every $i$
- Have $P(B \mid A_i)$, for every $i$

\[
P(B) = \sum_i P(A_i)P(B \mid A_i)
\]
Bayes’ rule

- Partition of sample space into $A_1, A_2, A_3$
- Have $P(A_i)$, for every $i$ — initial “beliefs”
- Have $P(B \mid A_i)$, for every $i$

revised “beliefs,” given that $B$ occurred:

$$P(A_i \mid B) = \frac{P(A_i)P(B \mid A_i)}{\sum_j P(A_j)P(B \mid A_j)}$$
Bayes’ rule and inference

- Thomas Bayes, presbyterian minister (c. 1701-1761)
- “Bayes’ theorem,” published posthumously
- systematic approach for incorporating new evidence

• Bayesian inference
  - initial beliefs $P(A_i)$ on possible causes of an observed event $B$
  - model of the world under each $A_i$: $P(B | A_i)$

\[ A_i \xrightarrow{\text{model}} P(B | A_i) \xrightarrow{\text{inference}} B \]

- draw conclusions about causes

\[ B \xrightarrow{\text{inference}} P(A_i | B) \xrightarrow{} A_i \]