LECTURE 4: Counting

Discrete uniform law
- Assume $\Omega$ consists of $n$ equally likely elements
- Assume $A$ consists of $k$ elements

Then: $P(A) = \frac{\text{number of elements of } A}{\text{number of elements of } \Omega} = \frac{k}{n}$

• Basic counting principle

• Applications
  - permutations
  - combinations
  - partitions
  - number of subsets
  - binomial probabilities
Basic counting principle

4 shirts
3 ties
2 jackets

Number of possible attires?

- $r$ stages
- $n_i$ choices at stage $i$

Number of choices is:
Basic counting principle examples

• Number of license plates with 2 letters followed by 3 digits:
  
  • ... if repetition is prohibited:

• Permutations: Number of ways of ordering $n$ elements:

• Number of subsets of $\{1, \ldots, n\}$:
Example

- Find the probability that:
  six rolls of a (six-sided) die all give different numbers.

(Assume all outcomes equally likely.)
Combinations

• Definition: \( \binom{n}{k} \): number of \( k \)-element subsets of a given \( n \)-element set

\[
\binom{n}{k} = \frac{n!}{k!(n-k)!}
\]

• Two ways of constructing an ordered sequence of \( k \) distinct items:
  - Choose the \( k \) items one at a time
  - Choose \( k \) items, then order them
\[
\binom{n}{k} = \frac{n!}{k!(n-k)!}
\]

\[
\binom{n}{n} =
\]

\[
\binom{n}{0} =
\]

\[
\sum_{k=0}^{n} \binom{n}{k} =
\]
Binomial coefficient \( \binom{n}{k} \) \rightarrow \text{Binomial probabilities}

- \( n \geq 1 \) independent coin tosses; \( P(H) = p \)
- \( P(HTTHHH) = \)
- \( P(\text{particular sequence}) = \)
- \( P(\text{particular } k-\text{head sequence}) \)

\[ P(k \text{ heads}) = \binom{n}{k} p^k (1 - p)^{n-k} \]
A coin tossing problem

- Given that there were 3 heads in 10 tosses, what is the probability that the first two tosses were heads?
  - event A: the first 2 tosses were heads
  - event B: 3 out of 10 tosses were heads

First solution:

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)} =$$

Assumptions:
- independence
- $P(H) = p$

$$P(k \text{ heads}) = \binom{n}{k} p^k (1 - p)^{n-k}$$
A coin tossing problem

- Given that there were 3 heads in 10 tosses, what is the probability that the first two tosses were heads?
  - event $A$: the first 2 tosses were heads
  - event $B$: 3 out of 10 tosses were heads

- Second solution: Conditional probability law (on $B$) is uniform

Assumptions:
- independence
- $P(H) = p$

$P(k \text{ heads}) = \binom{n}{k} p^k (1 - p)^{n-k}$
Partitions

- $n \geq 1$ distinct items; $r \geq 1$ persons
give $n_i$ items to person $i$
- here $n_1, \ldots, n_r$ are given nonnegative integers
- with $n_1 + \cdots + n_r = n$

- Ordering $n$ items:
  - Deal $n_i$ to each person $i$, and then order

\[
\text{number of partitions} = \frac{n!}{n_1! n_2! \cdots n_r!} \quad \text{(multinomial coefficient)}
\]
Example: 52-card deck, dealt (fairly) to four players. Find $P$(each player gets an ace)

- Outcomes are:
  - number of outcomes:

- Constructing an outcome with one ace for each person:
  - distribute the aces
  - distribute the remaining 48 cards

\[
\frac{4 \cdot 3 \cdot 2 \cdot \frac{48!}{12! 12! 12! 12!}}{52!} = \frac{13! 13! 13! 13!}{13! 13! 13! 13!}
\]

• Answer:
Example: 52-card deck, dealt (fairly) to four players. Find $P$(each player gets an ace)

A smart solution

Stack the deck, aces on top

Deal, one at a time, to available “slots”