LECTURE 4: Counting

Discrete uniform law
- Assume $\Omega$ consists of $n$ equally likely elements
- Assume $A$ consists of $k$ elements

Then: $P(A) = \frac{\text{number of elements of } A}{\text{number of elements of } \Omega} = \frac{k}{n}$

- Basic counting principle
- Applications
  - permutations
  - number of subsets
  - combinations
  - binomial probabilities
  - partitions
Basic counting principle

4 shirts
3 ties
2 jackets

Number of possible attires?

- \( r \) stages
- \( n_i \) choices at stage \( i \)

Number of choices is: \( n_1 \cdot n_2 \cdot \ldots \cdot n_r \).
Basic counting principle examples

- Number of license plates with 2 letters followed by 3 digits:
  \[ 26 \times 26 \times 10 \times 10 \times 10 \]
  - ... if repetition is prohibited: \[ 26 \times 25 \times 10 \times 9 \times 8 \]

- Permutations: Number of ways of ordering \( n \) elements:
  \[ n! = n \times (n-1) \times (n-2) \times \cdots \times 1 \]

- Number of subsets of \( \{1, \ldots, n\} \):
  \[ 2 \times 2 \times \cdots \times 2 = 2^n \]
Example

- Find the probability that:
  six rolls of a (six-sided) die all give different numbers.

(Assume all outcomes equally likely.)

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typical outcome \[ P(2,3,4,3,6,2) = \frac{1}{66} \]
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"element of \( A \): \( (2,3,4,1,6,5) = 6! \)
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\[
P(A) = \frac{\# \text{ in } A}{\# \text{ possible outcomes}} = \frac{6!}{6^6}.
\]
Combinations

- Definition:
  \[ \binom{n}{k} : \text{number of } k\text{-element subsets of a given } n\text{-element set} \]
  \[ \frac{n!}{k!(n-k)!} \]

- Two ways of constructing an ordered sequence of \( k \) distinct items:
  \( k = 0, 1, \ldots, n \)
  - Choose the \( k \) items one at a time
  - Choose \( k \) items, then order them
\[ \binom{n}{k} = \frac{n!}{k!(n-k)!} \]

\[ \binom{n}{n} = \frac{n!}{n!0!} = 1 \]

\[ \binom{n}{0} = \frac{0!}{0!n!} = 1 \]

\[ \sum_{k=0}^{n} \binom{n}{k} = \binom{n}{0} + \binom{n}{1} + \cdots + \binom{n}{n} = \# \text{ all subsets of } 2^n \]
Binomial coefficient \( \binom{n}{k} \) → Binomial probabilities

- \( n \geq 1 \) independent coin tosses; \( P(H) = p \)
- \( P(HTTHHH) = p(1-p)(1-p)p^2p^2 = p^4(1-p)^2 \)
- \( P(\text{particular sequence}) = p^\# \text{heads} (1-p)^\# \text{tails} \)
- \( P(\text{particular} k-\text{head sequence}) = p^k(1-p)^{n-k} \)

\[ P(k \text{ heads}) = p^k(1-p)^{n-k} \cdot \binom{n}{k} \]
A coin tossing problem

- Given that there were 3 heads in 10 tosses, what is the probability that the first two tosses were heads?
  - event $A$: the first 2 tosses were heads
  - event $B$: 3 out of 10 tosses were heads

- First solution:
  \[
  P(A \mid B) = \frac{P(A \cap B)}{P(B)} = \frac{P(H_1, H_2 \text{ and one } H \text{ in tosses } 3, \ldots, 10)}{P(B)}
  \]
  \[
  = \frac{p^2 \cdot \binom{8}{1} p^1 (1-p)^7}{\binom{10}{3} p^3 (1-p)^7} = \frac{\binom{8}{1}}{\binom{10}{3}} \approx \frac{8}{15}. \]

Assumptions:
- independence
- $P(H) = p$

\[
P(k \text{ heads}) = \binom{n}{k} p^k (1-p)^{n-k}
\]
A coin tossing problem

- Given that there were 3 heads in 10 tosses, what is the probability that the first two tosses were heads?
  - event $A$: the first 2 tosses were heads
  - event $B$: 3 out of 10 tosses were heads

- Second solution: Conditional probability law (on $B$) is uniform

\[
\Pr(k \text{ heads}) = \binom{n}{k} p^k (1-p)^{n-k}
\]

Assumptions:
- independence
- $\Pr(H) = p$

$$\frac{\text{# in } (A \cap B)}{\text{# in } B} = \frac{8}{\binom{10}{3}}.$$
Partitions

- \( n \geq 1 \) distinct items; \( r \geq 1 \) persons
  
  - give \( n_i \) items to person \( i \)
  - here \( n_1, \ldots, n_r \) are given nonnegative integers
  - with \( n_1 + \cdots + n_r = n \)

- Ordering \( n \) items: \( n! \)
  
  - Deal \( n_i \) to each person \( i \), and then order

\[
\binom{n_1! \cdot n_2! \cdots n_r!}{n} = n!
\]

\[
\begin{align*}
  r &= 2 \\
  n_1 &= k \\
  n_2 &= n - k
\end{align*}
\]

**Number of partitions**

\[
\text{number of partitions} = \frac{n!}{n_1! n_2! \cdots n_r!}
\]

(multinomial coefficient)
Example: 52-card deck, dealt (fairly) to four players. Find \( P(\text{each player gets an ace}) \)

- Outcomes are: partition equally likely
  - number of outcomes: \( \frac{52!}{13! \cdot 13! \cdot 13! \cdot 13!} \)

- Constructing an outcome with one ace for each person:
  - distribute the aces \( 4 \cdot 3 \cdot 2 \cdot 1 \)
  - distribute the remaining 48 cards \( \frac{48!}{12! \cdot 12! \cdot 12! \cdot 12!} \)

- Answer: \( \frac{4 \cdot 3 \cdot 2 \cdot 48!}{12! \cdot 12! \cdot 12! \cdot 12!} \frac{52!}{13! \cdot 13! \cdot 13! \cdot 13!} \)
Example: 52-card deck, dealt (fairly) to four players. Find $P$(each player gets an ace)

Stack the deck, aces on top

$$\frac{39}{51} \cdot \frac{26}{50} \cdot \frac{13}{49} = 0.105$$