LECTURE 7: Conditioning on a random variable; Independence of r.v.'s

- Conditional PMFs
  - Conditional expectations
  - Total expectation theorem
- Independence of r.v.'s
  - Expectation properties
  - Variance properties
- The variance of the binomial
- The hat problem: mean and variance
Conditional PMFs

\[ p_{X|A}(x \mid A) = P(X = x \mid A) \quad p_{X|Y}(x \mid y) = P(X = x \mid Y = y) \]

\[ p_{X|Y}(x \mid y) = \frac{p_{X,Y}(x, y)}{p_Y(y)} \]

defined for \( y \) such that \( p_Y(y) > 0 \)

\[ \sum_x p_{X|Y}(x \mid y) = 1 \]

\[ p_{X,Y}(x, y) = p_Y(y) p_{X|Y}(x \mid y) \]

\[ p_{X,Y}(x, y) = p_X(x) p_{Y|X}(y \mid x) \]
Conditional PMFs involving more than two r.v.'s

- Self-explanatory notation

\[ p_{X|Y,Z}(x \mid y, z) \]
\[ p_{X,Y|Z}(x, y \mid z) \]

- Multiplication rule

\[ P(A \cap B \cap C) = P(A) P(B \mid A) P(C \mid A \cap B) \]

\[ p_{X,Y,Z}(x, y, z) = p_X(x) p_{Y\mid X}(y \mid x) p_{Z\mid X,Y}(z \mid x, y) \]
Conditional expectation

\[ E[X] = \sum_x x p_X(x) \quad E[X | A] = \sum_x x p_{X|A}(x) \quad E[X | Y = y] = \sum_x x p_{X|Y}(x | y) \]

- Expected value rule

\[ E[g(X)] = \sum_x g(x) p_X(x) \quad E[g(X) | A] = \sum_x g(x) p_{X|A}(x) \]

\[ E[g(X) | Y = y] = \sum_x g(x) p_{X|Y}(x | y) \]
Total probability and expectation theorems

- $A_1, \ldots, A_n$: partition of $\Omega$

- $p_X(x) = P(A_1) p_{X|A_1}(x) + \cdots + P(A_n) p_{X|A_n}(x)$

- $p_X(x) = \sum_y p_Y(y) p_{X|Y}(x | y)$

- $E[X] = P(A_1) E[X | A_1] + \cdots + P(A_n) E[X | A_n]$

- Fine print:
  Also valid when $Y$ is a discrete r.v. that ranges over an infinite set, as long as $E[|X|] < \infty$
Independence

- of two events: \( P(A \cap B) = P(A) \cdot P(B) \)  
  \( P(A | B) = P(A) \)

- of a r.v. and an event: \( P(X = x \text{ and } A) = P(X = x) \cdot P(A), \text{ for all } x \)

- of two r.v.'s: \( P(X = x \text{ and } Y = y) = P(X = x) \cdot P(Y = y), \text{ for all } x, y \)  
  \( p_{X,Y}(x,y) = p_X(x) p_Y(y), \text{ for all } x, y \)

\( X, Y, Z \) are **independent** if: \( p_{X,Y,Z}(x,y,z) = p_X(x) p_Y(y) p_Z(z), \text{ for all } x, y, z \)
Example: independence and conditional independence

- Independent?

- What if we condition on $X \leq 2$ and $Y \geq 3$?
Independence and expectations

- In general: $E[g(X, Y)] \neq g(E[X], E[Y])$

- Exceptions:
  
  $E[aX + b] = aE[X] + b$  

If $X, Y$ are independent:  

$E[XY] = E[X]E[Y]$  

$g(X)$ and $h(Y)$ are also independent:  

$E[g(X)h(Y)] = E[g(X)] \cdot E[h(Y)]$
Independence and variances

- Always true: \( \text{var}(aX) = a^2 \text{var}(X) \) \quad \text{var}(X + a) = \text{var}(X) 

- In general: \( \text{var}(X + Y) \neq \text{var}(X) + \text{var}(Y) \)

If \( X, Y \) are independent: \( \text{var}(X + Y) = \text{var}(X) + \text{var}(Y) \)

- Examples:
  - If \( X = Y \): \( \text{var}(X + Y) = \)
  - If \( X = -Y \): \( \text{var}(X + Y) = \)
  - If \( X, Y \) independent: \( \text{var}(X - 3Y) = \)
Variance of the binomial

- \( X \): binomial with parameters \( n, p \)
  - number of successes in \( n \) independent trials

\[
X_i = \begin{cases} 
1 & \text{if } i\text{th trial is a success;} \\
0 & \text{otherwise}
\end{cases} \quad \text{(indicator variable)}
\]

\[
X = X_1 + \cdots + X_n
\]
The hat problem

- $n$ people throw their hats in a box and then pick one at random
  - All permutations equally likely
  - Equivalent to picking one hat at a time

- $X$: number of people who get their own hat

  - Find $E[X]$

  $$X_i = \begin{cases} 1, & \text{if } i \text{ selects own hat} \\ 0, & \text{otherwise.} \end{cases}$$

  $$X = X_1 + X_2 + \cdots + X_n$$

- $E[X_i] =$
The variance in the hat problem

• \( X \): number of people who get their own hat
  
  - Find \( \text{var}(X) \)

  \[
  X_i = \begin{cases} 
  1, & \text{if } i \text{ selects own hat} \\
  0, & \text{otherwise.}
  \end{cases}
  \]

  \[ X = X_1 + X_2 + \cdots + X_n \]

  \[
  X^2 = \sum_i X_i^2 + \sum_{i,j:i \neq j} X_i X_j
  \]

• \( \text{var}(X) = E[X^2] - (E[X])^2 \)

• \( E[X_i^2] = \)

• For \( i \neq j \): \( E[X_i X_j] = \)

• \( E[X^2] = \)