LECTURE 13: Conditional expectation and variance revisited;

Application: Sum of a random number of independent r.v.’s

- A more abstract version of the conditional expectation
  - view it as a random variable
  - the law of iterated expectations

- A more abstract version of the conditional variance
  - view it as a random variable
  - the law of total variance

- Sum of a random number of independent r.v.’s
  - mean
  - variance
Conditional expectation as a random variable

- Function $h$
  
  e.g., $h(x) = x^2$, for all $x$

- Random variable $X$; what is $h(X)$?
  
  $h(X)$ is the r.v. that takes the value $x^2$, if $X$ happens to take the value $x$

- $g(y) = \mathbb{E}[X \mid Y = y] = \sum_x x p_{X \mid Y}(x \mid y)$
  
  (integral in continuous case)

- $g(y)$: is the r.v. that takes the value $\mathbb{E}[X \mid Y = y]$, if $Y$ happens to take the value $y$

- Remarks:
  
  - It is a function of $Y$
  - It is a random variable
  - Has a distribution, mean, variance, etc.

**Definition:** $\mathbb{E}[X \mid Y] = g(Y)$
The mean of $E[X | Y]$: Law of iterated expectations

- $g(y) = E[X | Y = y]$

$E[X | Y] = g(y)$

$E[E[X | Y]] = E[g(y)]$

$= \sum_y g(y) P_Y(y)$

$= \sum_y E[X | Y = y] P_Y(y)$

$= E[X]$
Stick-breaking example

- Stick example: stick of length $\ell$
  break at uniformly chosen point $Y$

  break what is left at uniformly chosen point $X$

- $E[X | Y = y] = \frac{y}{2}$

- $E[X | Y] = \frac{y}{2}$

$E[X] = E\left[E[X | Y]\right] = E\left[\frac{Y}{2}\right] = \frac{1}{2} \cdot E[Y] = \frac{1}{2} \cdot \frac{\ell}{2} = \frac{\ell}{4}$
Forecast revisions

- Suppose forecasts are made by calculating expected value, given any available information.

- $X$: February sales

- Forecast in the beginning of the year: $E[X]$.

- End of January: will get new information, value $y$ of $Y$.

  Revised forecast: $E[X|Y=y]$.

- Law of iterated expectations:

  $E[E[X|Y]] = E[X] = \text{original forecast}$.
The conditional variance as a random variable

\[
\text{var}(X) = E[ (X - E[X])^2 ]
\]

\[
\text{var}(X \mid Y = y) = E[ (X - E[X \mid Y = y])^2 \mid Y = y ]
\]

\[
\text{var}(X \mid Y) \text{ is the r.v. that takes the value var}(X \mid Y = y), \text{ when } Y = y
\]

- Example: \( X \) uniform on \([0, Y]\)

\[
\begin{align*}
\text{var}(X \mid Y = y) &= \frac{y^2}{12} \\
\text{var}(X \mid Y) &= \frac{Y^2}{12}
\end{align*}
\]

Law of total variance: \( \text{var}(X) = E[ \text{var}(X \mid Y) ] + \text{var}(E[X \mid Y]) \)
Derivation of the law of total variance

\[ \text{var}(X) = E[\text{var}(X \mid Y)] + \text{var}(E[X \mid Y]) \]

• \( \text{var}(X) = E[X^2] - (E[X])^2 \)

\[ \text{var}(X \mid Y = y) = E[x^2 \mid Y = y] - \left( E[x \mid Y = y] \right)^2 \text{ for all } y \]

\[ \text{var}(X \mid Y) = E[x^2 \mid Y] - \left( E[x \mid Y] \right)^2 \]

\[ E[\text{var}(X \mid Y)] = E[x^2] - E\left[ \left( E[x \mid Y] \right)^2 \right] \]

\[ + \text{var}(E[X \mid Y]) = E\left[ (E[x \mid Y])^2 \right] - \left( E[E[x \mid Y]] \right)^2 \left( E[x] \right)^2 \]
A simple example

\[ \text{var}(X) = \mathbb{E}[\text{var}(X | Y)] + \text{var}(\mathbb{E}[X | Y]) \]

\[ = \frac{5}{24} + \frac{9}{16} = \frac{37}{48} \]

\[ \text{var}(X | Y) = \begin{cases} \frac{1}{12} & \text{if } Y = 1 \\ \frac{1}{12} & \text{if } Y = 2 \end{cases} \]

\[ \mathbb{E}[\text{var}(X | Y)] = \frac{1}{2} \cdot \frac{1}{12} + \frac{1}{2} \cdot \frac{4}{12} = \frac{5}{24} \]

\[ \text{var}(\mathbb{E}[X | Y]) = \frac{1}{2} \left( \frac{1}{2} - \frac{5}{4} \right)^2 + \frac{1}{2} \left( 2 - \frac{5}{4} \right)^2 = \frac{9}{16} \]
Section means and variances

- Two sections of a class: $y = 1$ (10 students); $y = 2$ (20 students)
  $x_i$: score of student $i$
- Experiment: pick a student at random (uniformly) random variables: $X$ and $Y$
- Data: $y = 1: \frac{1}{10} \sum_{i=1}^{10} x_i = 90$; $y = 2: \frac{1}{20} \sum_{i=11}^{30} x_i = 60$

- $E[X] = \frac{1}{30} \sum_{i=1}^{30} x_i = \frac{1}{30} (90 \cdot 10 + 60 \cdot 20) = 70$

  $E[X | Y = 1] = 90$  \hspace{1cm} $E[X | Y] = 70$

  $E[X | Y = 2] = 60$

- $E[E[X | Y]] = \frac{1}{3} \cdot 90 + \frac{2}{3} \cdot 60 = 70$
Section means and variances (ctd.)

\[ \mathbb{E}[X \mid Y] = \begin{cases} 
90, & \text{w.p. } 1/3 \\
60, & \text{w.p. } 2/3
\end{cases} \]

\[ \mathbb{E}[\mathbb{E}[X \mid Y]] = 70 = \mathbb{E}[X] \]

\[ \text{var}(\mathbb{E}[X \mid Y]) = \frac{1}{3}(90-70)^2 + \frac{2}{3}(60-70)^2 = 200 \]

- More data:
  \[ \frac{1}{10} \sum_{i=1}^{10} (x_i - 90)^2 = 10 \]
  \[ \frac{1}{20} \sum_{i=11}^{30} (x_i - 60)^2 = 20 \]

\[ \text{var}(X \mid Y = 1) = 10 \]

\[ \text{var}(X \mid Y) = \frac{1}{3} \cdot 10 + \frac{2}{3} \cdot 20 = \frac{50}{3} \]

\[ \text{var}(X) = \mathbb{E}[\text{var}(X \mid Y)] + \text{var}(\mathbb{E}[X \mid Y]) = \frac{50}{3} + 200 \]

\[ \text{var}(X) = (\text{average variability within sections}) + (\text{variability between sections}) \]
Sum of a random number of independent r.v.'s

- $N$: number of stores visited
  ($N$ is a nonnegative integer r.v.)
- Let $Y = X_1 + \cdots + X_N$
- $X_i$: money spent in store $i$
  - $X_i$ independent, identically distributed
  - independent of $N$

\[
E[Y \mid N = n] = E[X_1 + \cdots + X_n \mid N = n] = E[X_1 + \cdots + X_n \mid N = n] = E[X_1 + \cdots + X_n] = nE[X]
\]

- Total expectation theorem:
  \[
  E[Y] = \sum_n p_N(n) E[Y \mid N = n] = \sum_n p_N(n) nE[X] = E[N]E[X]
  \]

- Law of iterated expectations:
  \[
  E[Y] = E[E[Y \mid N]] = E[NE[X]] = E[N]E[X]
  \]
Variance of sum of a random number of independent r.v.'s

\[ Y = X_1 + \cdots + X_N \]

\[ \text{var}(Y) = E[\text{var}(Y | N)] + \text{var}(E[Y | N]) \]

\[ \text{var}(Y) = E[N] \text{var}(X) + (E[X])^2 \text{var}(N) \]

- \( E[Y | N] = N E[X] \)

\[ \text{var}(E[Y | N]) = \text{var}(NE[X]) = (E[X])^2 \text{var}(N) \]

- \( \text{var}(Y | N = n) = \text{var}(X_1 + \cdots + X_n | N = n) = \text{var}(X_1 + \cdots + X_n) = n \text{var}(X) \)

- \( \text{var}(Y | N) = N \text{var}(X) \)

- \( E[\text{var}(Y | N)] = E[N \text{var}(X)] = E[N] \text{var}(X) \)