LECTURE 21: The Bernoulli process

- Definition of Bernoulli process
- Stochastic processes
- Basic properties (memorylessness)
- The time of the $k$th success/arrival
- Distribution of interarrival times
- Merging and splitting
- Poisson approximation
The Bernoulli process

- A sequence of independent Bernoulli trials, $X_i$
- At each trial, $i$:
  \[ P(X_i = 1) = P(\text{success at the } i\text{th trial}) = p \]
  \[ P(X_i = 0) = P(\text{failure at the } i\text{th trial}) = 1 - p \]
- Key assumptions:
  - Independence
  - Time-homogeneity
- Model of:
  - Sequence of lottery wins/losses
  - Arrivals (each second) to a bank
  - Arrivals (at each time slot) to server
  - ...

- Jacob Bernoulli (1655–1705)
Stochastic processes

- First view: sequence of random variables $X_1, X_2, \ldots$

  Interested in: $E[X_i] = p$, $\text{var}(X_i) = p(1-p)$, $p_{X_i}(x) = \begin{cases} p & x = 1 \\ 1-p & x = 0 \end{cases}$

  $p_{X_1,\ldots,X_n}(x_1,\ldots,x_n) = p_{X_1}(x_1) \cdots p_{X_n}(x_n)$ for all $n$

- Second view – sample space:

  $\Omega = \text{set of infinite sequences of 0's and 1's}$

- Example (for Bernoulli process):

  $P(X_i = 1 \text{ for all } i) = 0 \quad (p < 1)$

  $\leq P(X_1 = 1, \ldots, X_n = 1) = p^n$, for all $n$
Number of successes/arrivals $S$ in $n$ time slots

- $S = X_1 + \cdots + X_n$
- $P(S = k) = \binom{n}{k} p^k (1-p)^{n-k}$ for $k = 0, \ldots, n$
- $E[S] = np$
- $\text{var}(S) = np(1-p)$
Time until the first success/arrival

- \( T_1 = \min \{ i : X_i = 1 \} \)

- \( P(T_1 = k) = p \left( \underbrace{0 \ldots 0}_{k-1} 1 \right) = (1-p)^{k-1} p \quad k = 1, 2, \ldots \)

- \( E[T_1] = \frac{1}{p} \)

- \( \text{var}(T_1) = \frac{1-p}{p^2} \)
Independence, memorylessness, and fresh-start properties

\[ \{X, Y \} \sim \text{Ber}(p) \]

\[ Y_1 = X_{n+1} \quad \{Y_2\} \]

\[ Y_2 = X_{n+2} \quad i = 1, 2, \ldots \]

1. \( \{Y_1, Y_2\} \) independent of \( X_1, \ldots, X_n \)

2. \( \text{Ber}(p) \)

- Fresh-start after time \( n \)

- Fresh-start after time \( T_1 \)
Independence, memorylessness, and fresh-start properties

- Fresh-start after a random time $N$?
  - $N = \text{time of 3rd success}$
  - $N = \text{first time that 3 successes in a row have been observed}$
  - $N = \text{the time just before the first occurrence of 1,1,1}$

The process $X_{N+1}, X_{N+2}, \ldots$ is:
- a Bernoulli process (as long as $N$ is determined “causally”)
- independent of $N, X_1, \ldots, X_N$
The distribution of busy periods

- At each slot, a server is busy or idle (Bernoulli process)

- First busy period: \( \text{Geo}(1-p) \)
  - starts with first busy slot
  - ends just before the first subsequent idle slot
Time of the $k$th success/arrival

- $Y_k = \text{time of } k\text{th arrival}$

- $T_k = k\text{th inter-arrival time} = Y_k - Y_{k-1} \quad (k \geq 2)$

- The process starts fresh after time $T_1$

- $T_2$ is independent of $T_1$; Geometric($p$); etc.
Time of the $k$th success/arrival

$$P(Y_k = t)$$

$= P(\text{k-1 arrivals in time } t-1)$

$P(\text{arrival at time } t)$

$= \binom{t-1}{k-1} p^{k-1} (1-p)^{t-k} \cdot p$

$= \binom{t-1}{k-1} p^k (1-p)^{t-k}$

$Y_k = T_1 + \cdots + T_k$

the $T_i$ are i.i.d., Geometric($p$)

$E[Y_k] = \frac{k}{p}$

$\text{var}(Y_k) = \frac{k(1-p)}{p^2}$

$p_{Y_k}(t) = \binom{t-1}{k-1} p^k (1-p)^{t-k}$,

$t = k, k+1, \ldots$
Merging of independent Bernoulli processes

$X_t \sim \text{Bernoulli}(p)$  

merged process $Z_t \sim \text{Bernoulli}(p + q - pq)$ (collisions are counted as one arrival)

$Y_t \sim \text{Bernoulli}(q)$  

$Z_t = g(X_t, Y_t) \quad (Z_1, \ldots, Z_t)$

$Z_{t+1} = g(X_{t+1}, Y_{t+1}) \quad 1 - (1-p)(1-q)$

$P(\text{arrival in first process} \mid \text{arrival}) = \frac{p}{p + q - pq}$
Splitting of a Bernoulli process

- Split successes into two streams, using independent flips of a coin with bias $q$
  - assume that coin flips are independent from the original Bernoulli process

Bernoulli($pq$)  \[ \begin{array}{ccccccc} \times & | & | & | & \times & \cdots & \end{array} \] time

Bernoulli($p$)  \[ \begin{array}{cccccccc} \times & | & \times & | & \times & \cdots \end{array} \] time

Bernoulli($p(1-q)$)  \[ \begin{array}{cccccccc} | & | & \times & | & \times & \cdots \end{array} \] time

- Are the two resulting streams independent? **No**
Poisson approximation to binomial

- Interesting regime: large \( n \), small \( p \), moderate \( \lambda = np \)

- Number of arrivals \( S \) in \( n \) slots:
  \[
  p_S(k) = \frac{n!}{(n-k)!k!} \cdot p^k (1-p)^{n-k}, \quad k = 0, \ldots, n
  \]

For fixed \( k = 0, 1, \ldots, \)
  \[
  p_S(k) \to \frac{\lambda^k}{k!} e^{-\lambda},
  \]

- Fact: \( \lim_{n \to \infty} (1 - \lambda/n)^n = e^{-\lambda} \)
The following may not correspond to a particular course on MIT OpenCourseWare, but has been provided by the author as an individual learning resource.

For information about citing these materials or our Terms of Use, visit: https://ocw.mit.edu/terms.