LECTURE 23: More on the Poisson process

- The sum of independent Poisson r.v.s
- Merging and splitting
- Random incidence
The sum of independent Poisson random variables

- Poisson process of rate $\lambda = 1$

- Consecutive intervals of length $\mu$ and $\nu$

- Numbers of arrivals during these intervals: $M$ and $N$
  - $M$: Independent?
  - $N$: $M + N$:

The sum of independent Poisson random variables, with means/parameters $\mu$ and $\nu$, is Poisson with mean/parameter $\mu + \nu$
## Merging of independent Poisson processes

<table>
<thead>
<tr>
<th></th>
<th>$1 - \lambda_1 \delta$</th>
<th>$\lambda_1 \delta$</th>
<th>$O(\delta^2)$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>0</td>
<td>1</td>
<td>$\geq 2$</td>
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<tr>
<td>$1 - \lambda_2 \delta$</td>
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<td>$\lambda_2 \delta$</td>
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<tr>
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<td>$\geq 2$</td>
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**Diagram:**
- Red bulb flashes (Poisson).
- Green bulb flashes (Poisson).

Merged process: \( \text{Poisson}(\lambda_1 + \lambda_2) \)
Where is an arrival of the merged process coming from?

\[ P(\text{Red} \mid \text{arrival at time } t) = \]

\[
\begin{array}{c|ccc}
   & 1 - \lambda_1 \delta & \lambda_1 \delta & O(\delta^2) \\
\hline
1 - \lambda_2 \delta & 0 & 1 & \geq 2 \\
\lambda_2 \delta & 1 - (\lambda_1 + \lambda_2) \delta & \lambda_1 \delta & O(\delta^2) \\
O(\delta^2) & \geq 2 & & \\
\end{array}
\]

\[ P(\text{kth arrival is Red}) = \]

- **Independence** for different arrivals

\[ P(4 \text{ out of first 10 arrivals are Red}) = \]
The time the first (or the last) light bulb burns out

- Three lightbulbs
  - independent lifetimes $X, Y, Z$; exponential($\lambda$)
- Find expected time until first burnout

- $X, Y, Z$: first arrivals in independent Poisson processes
- Merged process:
  - $\min\{X, Y, Z\}$: 1st arrival in merged process
The time the first (or the last) light bulb burns out

- Three lightbulbs
  - independent lifetimes $X$, $Y$, $Z$; exponential($\lambda$)
- Find expected time until all burn out
Splitting of a Poisson process

- Split arrivals into two streams, using independent coin flips of a coin with bias $q$
  
  - assume that coin flips are independent from the original Poisson process

\[ \cdots \quad \text{time} \]

\[ \times \times \times \times \quad \cdots \quad \text{time} \]

\[ \cdots \quad \text{time} \]

Resulting streams are Poisson, rates $\lambda q$, $\lambda (1 - q)$

- Are the two resulting streams independent?
  
  Surprisingly, yes!
“Random incidence” in the Poisson process

- Poisson process that has been running forever

- Believe that $\lambda = 4$/hour, so that $E[T_k] =$

- Show up at some time and measure interarrival time
  - do it many times, average results, see something around 30 mins! Why?
“Random incidence” in the Poisson process — analysis

- Arrive at time \( t^* \)
- \( U \): last arrival time
- \( V \): next arrival time
- \( V - U = \)
- \( E[V - U] = \)
- \( V - U \): interarrival time you see, versus \( k \)th interarrival time
Random incidence “paradox” is not special to the Poisson process

- **Example:** interarrival times, i.i.d., equally likely to be 5 or 10 minutes

  expected value of $k$th interarrival time:

- you show up at a “random time”

  $P(\text{arrive during a 5-minute interarrival interval}) = $

  expected length of interarrival interval during which you arrive =

- Calculation generalizes to “renewal processes:”
  i.i.d. interarrival times, from some general distribution

- “Sampling method” matters
Different sampling methods can give different results

- Average family size?
  - look at a “random” family (uniformly chosen)
  - look at a “random” person’s (uniformly chosen) family

- Average bus occupancy?
  - look at a “random” bus (uniformly chosen)
  - look at a “random” passenger’s bus

- Average class size?
Resource: Introduction to Probability
John Tsitsiklis and Patrick Jaillet

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