Markov processes – II

- review and some warm-up
  - definitions, Markov property
  - calculating the probabilities of trajectories
- steady-state behavior
  - recurrent states, transient states, recurrent classes
  - periodic states
  - convergence theorem
  - balance equations
- birth-death processes
review

- discrete time, discrete state space, time-homogeneous
  - transition probabilities
  - Markov property

- \( r_{ij}(n) = P(X_n = j \mid X_0 = i) \)
  = \( P(X_{n+s} = j \mid X_s = i) \)

- key recursion:

\[
    r_{ij}(n) = \sum_{k=1}^{m} r_{ik}(n-1)p_{kj}
\]
warmup

\[ P(X_1 = 2, X_2 = 6, X_3 = 7 \mid X_0 = 1) = \]

\[ P(X_4 = 7 \mid X_0 = 2) = \]
review: recurrent and transient states

- state i is **recurrent** if “starting from i, and from wherever you can go, there is a way of returning to i”
- if not recurrent, called **transient**

- **recurrent class**: a collection of recurrent states communicating only between each other
periodic states in a recurrent class

The states in a recurrent class are periodic if they can be grouped into $d > 1$ groups so that all transitions from one group lead to the next group.
steady-state probabilities

- does \( r_{ij}(n) = \Pr(X_n = j \mid X_0 = i) \) converge to some \( \pi_j \)?
- theorem: yes, if:
  - recurrent states are all in a single class, and
  - single recurrent class is not periodic

- assuming “yes”, start from key recursion \( r_{ij}(n) = \sum_k r_{ik}(n-1)p_{kj} \)
  - take the limit as \( n \to \infty \)
  \[ \pi_j = \sum_k \pi_k p_{kj} \]
  - need also: \( \sum_j \pi_j = 1 \)
example

\[ \pi_j = \sum_{k} \pi_k p_{kj} \]
visit frequency interpretation

- balance equations
  \[ \pi_j = \sum_k \pi_k p_{kj} \]

- (long run) frequency of being in \( j \): \( \pi_j \)

- frequency of transitions \( 1 \rightarrow j \): \( \pi_1 p_{1j} \)

- frequency of transitions into \( j \): \( \sum_k \pi_k p_{kj} \)
birth-death processes

\[ \pi_i p_i = \pi_{i+1} q_{i+1} \]
birth-death processes II

special case: \( p_i = p \) and \( q_i = q \) for all \( i \)

\[
\rho = \frac{p}{q} \quad \pi_{i+1} = \pi_i \frac{p}{q} = \pi_i \rho \\
\pi_i = \pi_0 \rho^i \quad i = 0, 1, \ldots, m
\]

- assume \( p = q \)

- assume \( p < q \) and \( m \approx \infty \)

\[
\pi_0 = 1 - \rho \quad E[X_n] = \frac{\rho}{1 - \rho} \quad \text{(in steady-state)}
\]