Markov processes – II

• review and some warm-up
  – definitions, Markov property
  – calculating the probabilities of trajectories

• steady-state behavior
  – recurrent states, transient states, recurrent classes
  – periodic states
  – convergence theorem
  – balance equations

• birth-death processes
review

- discrete time, discrete state space, time-homogeneous
  - transition probabilities
  - Markov property

\[
\begin{align*}
\pi_{ij}(n) &= P(X_n = j \mid X_0 = i) \\
&= P(X_{n+s} = j \mid X_s = i) \\
&= \pi_{ij} \\
&= \sum_{k=1}^{m} \pi_{ik}(n-1) \pi_{kj}
\end{align*}
\]

key recursion:
warmup

"multiplicative rule"

\[
P(B \land C \land D \mid A) = P(B \mid A) \times P(C \mid A \land B) \times P(D \mid A \land B \land C)
\]

\[
P(X_1 = 2, X_2 = 6, X_3 = 7 \mid X_0 = 1) = P(X_1 = 2 \mid X_0 = 1) \times P(X_2 = 6 \mid X_0 = 1, X_1 = 2) \times P(X_3 = 7 \mid X_0 = 1, X_1 = 2, X_2 = 6)
\]

\[
P(X_4 = 7 \mid X_0 = 2) = P_{6-7} \times P_{6-7} \times P_{6-7} \times P_{6-7} + P_{1-2} \times P_{1-2} \times P_{6-7} \times P_{6-7}
\]

\[
= \frac{m^2}{m \times m^2}
\]
review: recurrent and transient states

- state $i$ is recurrent if “starting from $i$, and from wherever you can go, there is a way of returning to $i$”
- if not recurrent, called transient

- recurrent class: a collection of recurrent states communicating only between each other
The states in a recurrent class are periodic if they can be grouped into $d > 1$ groups so that all transitions from one group lead to the next group.
steady-state probabilities

Does $r_{ij}(n) = P(X_n = j | X_0 = i)$ converge to some $\pi_j$?

Theorem: Yes, if:
- Recurrent states are all in a single class, and
- Single recurrent class is not periodic

Assuming "yes", start from key recursion $r_{ij}(n) = \sum_{k=1}^{m} r_{ik}(n-1)p_{kj}$
- Take the limit as $n \to \infty$
- Need also: $\sum_{j=1}^{m} \pi_j = 1$ and unique solution

$m$ equations, $m$ unknowns, $T_{ij}$'s
example

\begin{align*}
\pi_j &= \sum_{k=1}^{m} \pi_k P_{kj}, \quad j=1, \ldots, m \\
m &= 2 \\
\pi_1 &= \pi_1 \times 0.5 + \pi_2 \times 0.2 \quad \text{same} \\
\pi_2 &= \pi_1 \times 0.5 + \pi_2 \times 0.8 \\
\pi_1 \times \frac{1}{2} &= \pi_2 \times \frac{1}{5} \\
\pi_1 + \pi_2 &= 1 \\
\frac{\pi_1}{\pi_2} &= \frac{2}{5} \quad \frac{\pi_1}{\pi_2} = \frac{2}{5} \\

\end{align*}
visit frequency interpretation

• balance equations

\[ \pi_j = \sum_k \pi_k p_{kj} \]

• (long run) frequency of being in \( j \):

\[ \pi_j \]

• frequency of transitions \( 1 \rightarrow j \):

\[ \pi_1 p_{1j} \]

• frequency of transitions into \( j \):

\[ \sum_k \pi_k p_{kj} \]
birth-death processes

\[ \pi_{i+1} = \pi_i + 1 \cdot q_i + 1 \]

\[ \prod_{i=0}^{\infty} = 1 \Rightarrow \prod_{0}^{\infty} = \prod_{0}^{\infty} + \prod_{0}^{\infty} \cdot \frac{p_i}{q_i} + \prod_{0}^{\infty} \cdot \frac{p_i}{q_i} \cdot \frac{p_i}{q_i} + \cdots = 1 \]
birth-death processes

I

\[ \pi_{i+1} = \pi_i \frac{p_i}{q_{i+1}}, \quad i = 0, \ldots, m \]

\[ \sum_{j} \pi_j = 1 \]

special case: \( p_i = p \) and \( q_i = q \) for all \( i \)

\[ \rho = \frac{p}{q} \]

\[ \pi_i + 1 = \pi_i \frac{p}{q} = \pi_i \rho \]

\[ \pi_i = \pi_0 \rho^i, \quad i = 0, 1, \ldots, m \]

• assume \( p = q \)

\[ \Rightarrow \pi_i = \pi_0, \quad i = 0, \ldots, m \]

\[ \pi_0 \left[ 1 + \rho^2 + \cdots + \rho^m \right] = 1 \]

• assume \( p < q \) and \( m \approx \infty \)

\[ \pi_0 = 1 - \rho \]

\[ \frac{E[X_n]}{\pi_0} = \frac{\rho}{1 - \rho} \] (in steady-state)

\[ \pi_i = \pi_0 \rho^i = \left( \frac{1}{1 - \rho} \right) \rho^i, \ldots \]