

## MITOCW | linear\_approximations

N<sub>2</sub>O<sub>5</sub> is a reagent that was used to synthesize explosives like TNT. Unfortunately, N<sub>2</sub>O<sub>5</sub> decomposes relatively quickly at or above room temperature, so if you accidentally leave a bottle of it out on the counter, you could be in trouble! In this video, we'll approximate a solution to the decomposition rate equation to figure out whether a batch of N<sub>2</sub>O<sub>5</sub> that you left at room temperature can still be used. We'll also determine under what conditions our approximation is valid. This video is part of the Linearity video series. Many complex systems are modeled or approximated linearly because of the mathematical advantages. Hi, my name is Ben Brubaker, and I'm a professor in the Department of Mathematics at MIT. Today we'll be talking about linear approximations. In mathematical terms, this is just another name for the tangent line to a function. But the name suggests more. Linear approximations can be used to simplify mathematical models that are not analytically solvable. The approximated model will have a solution that is only acceptable under suitable conditions. However, it can still illuminate the behavior of the system within a certain acceptable range. Before watching this video, you should know the definition of the derivative, and how to write the equation of a line with a given slope that passes through a given point. After watching this video, you should be able to recognize the linear approximation of a function as the tangent line to the function, apply linear approximations to solve simple differential equations, and explain the limitations of linear approximations both mathematically and graphically. Let's begin by defining the linear approximation. Recall that if a function is differentiable at a point  $c$ , then when we zoom in on the point  $c$ , the function begins to look more and more like a line. This only works when the function is "smooth"—it doesn't have any kinks, corners, or discontinuities. Given a function  $f(x)$ , which is differentiable at the point  $c$ , we define the linear approximation to be the tangent line to the function at  $c$ . This is a line whose slope equals  $f'(c)$ . As an example, let's look at the following cubic equation:  $f(x) = \frac{1}{30}x^3 - 2x + 5$ . This function is differentiable everywhere, and in particular is differentiable at the point  $x=3$ . Find the equation of the tangent line to this cubic equation at  $x=3$  for yourself. We've also found the equation for the tangent line. To do this, we found the value of the function at 3, which we found to be  $\frac{4}{5}$ . And then we computed the derivative and evaluated it at  $x=3$ , which we found to be  $\frac{7}{6}$ . We'll write the tangent line as  $T_3(x)$  to remind us that we are finding the tangent to our function at the point 3. Then the equation for this line can be written as  $T_3(x) = \frac{4}{5} + \frac{7}{6}(x-3)$ . Let's take a look at the graphs of  $f(x)$  and  $T_3(x)$ . The function  $f(x)$  is graphed in red. The further we zoom into the graph at  $x=3$ , the more it begins to resemble a straight line. The slope of the line approaches the value of the slope of the tangent line at 3, which is drawn in blue. The tangent line certainly seems to be a good approximation to our function when we are close to  $x=3$ . But what about when we zoom out? Is it still a good approximation? Let's give a mathematical justification that the tangent line is a good approximation. Our function is differentiable at a point, say  $x=3$ . This is equivalent to the statement that when  $x$  is near 3, the slope of the secant line is approximately equal to the slope of the tangent line. That is, for  $x$  near 3  $\frac{f(x)-f(3)}{x-3}$  is approximately equal to the value of the derivative at 3. In order to see why these statements are equivalent, we need explain what we mean by "approximately equal to". This means that we can make the difference between these two sides as small as any error bound we choose, provided that we choose  $x$  close enough to 3. And this is exactly the definition of differentiability at the point 3. To relate this to linear approximations, we use a bit of algebra. First we multiply both sides by  $(x-3)$ , and we get  $f(x) - f(3)$  approximately equal to  $f'(3)(x-3)$ . Adding  $f(3)$  to both sides, we get that  $f(x)$  is approximately equal to  $f(3) + f'(3)(x-3)$ . And this right hand side is the equation for the tangent line at 3. So indeed the tangent line closely approximates the function as long as  $x$  is close to 3. Here we see the tangent line drawn in blue, the function drawn in red. The graphs demonstrate our mathematical proof that the tangent line is a good approximation near  $x=3$ , but the further we get from  $x=3$  the worse the approximation seems. Now you might see that the function and tangent line intersect at  $x=-9$ . Is the tangent line a good approximation for the function near  $x=-9$ ? No, it isn't. If we move a small distance along the tangent line away from  $x=-9$ , this does not model the behavior of the function near  $x=-9$ . This property is extremely important in applications—no measurement or observed quantity is ever given exactly. A1 The linearization of a function that is differentiable at  $x=c$  is just another name for the tangent line through  $c$ . The A2 key properties of this tangent line are that it shares the same value as the function and the same first derivative of the function at  $x=c$ . And A3 we've seen through both graphical intuition and the definition of the derivative that this linearization closely approximates the function for points  $x$  sufficiently close to  $c$ . Let's see how we can apply a linear approximation to simplify a problem. Let's suppose that you are participating in undergraduate research in a chemistry lab. You have been using a .01 molar solution of N<sub>2</sub>O<sub>5</sub>. You bring it out of the refrigerator at 9:00 Monday morning, and promptly forget about it, leaving it on the counter in the lab, which is 25 degrees Celsius. When you realize you left the solution out, 1 hour has passed. You panic. The problem is that N<sub>2</sub>O<sub>5</sub> decomposes into NO<sub>2</sub> and O<sub>2</sub> at room temperature. If the molarity of the solution has changed significantly, it might ruin your experiments. Experiments have shown that the rate of decomposition follows first order kinetics. This means that the instantaneous rate of change in concentration of N<sub>2</sub>O<sub>5</sub> is proportional to the concentration of N<sub>2</sub>O<sub>5</sub>. The constant of proportionality,  $k$ , has been found experimentally to be 1.72 times ten to the negative 5 inverse seconds. Because you are panicking, you are having a hard time solving this differential equation. Instead, find the approximate decomposition in the N<sub>2</sub>O<sub>5</sub> solution after 1 hour using a linear approximation at time  $t=0$ . Note that because the reaction constant has units of inverse seconds, the variable  $t$  must have units of seconds. In order to produce a linear approximation at  $t=0$ , we need two ingredients: the concentration at time 0, and the value of the derivative at time 0. The initial concentration was .01 molar. To find the value of the derivative, we use our differential equation, and we find that the value of the derivative at time  $t=0$  is just minus  $k$  times the initial concentration at  $t=0$ , or negative  $k$  times .01 molar. Remember that  $k$  has units of inverse seconds. Putting this together into the formula for the tangent line at zero, we see that the linear approximation, is given by  $T_0(t) = (.01 \text{ molar}) \times (1 - kt)$ . Now we are interested in approximating the value at 1 hour, or 3600 seconds. Plugging in the value for  $k$ , we

find that to two significant figures  $T_0(3600) = .0094$  Molar. So the molarity is not so different. Phew!! Now you are feeling relieved, so when you look back at the reaction rate, you realize that in fact, a first order reaction rate is telling you that the concentration is some function of time whose derivative is proportional to itself. The function with this property is the exponential function! So the concentration function is equal to some constant times  $e$  to the negative  $kt$ . The constant must be the initial concentration, which is .01 Molar, so  $[N_2O_5] = (.01)e^{-kt}$ . Evaluating this expression at time  $t=3600$ , we find that the exact concentration is also .0094 Molar to two significant figures. To be more precise, we can look at the error, which is the absolute value of the exact solution minus the approximate solution all divided by the exact solution. The error at time  $t=3600$  seconds is .002 or .2%. This is really quite good. Generally speaking, we might consider an error of less than 5% to be acceptable. So our linear approximation was definitely within the acceptable range 1 hour later. But now it's Tuesday. Today you leave the .01 Molar  $N_2O_5$  on the counter for a full 10 hours. Oops. We know that the exact solution is an exponential decay. So do you think that the linear approximation will be an over estimate, an under estimate, or equal to the exact solution after 10 hours? [pause] The linear approximation of the concentration after 10 hours at room temperature is  $T_0(36000) = .0038$  Molar. The exact solution of the concentration after 10 hours is .0054 Molar. Comparing these two solutions, we see that linear approximation is an under estimate as we would expect, with error is .29 or 29%. So a linear approximation is definitely not acceptable for this time range. But now you begin to wonder, for what times is the linear approximation at  $t=0$  within 5% error of the exact solution? [pause] Because we are only interested in positive time values, you should have found that the approximation is acceptable for times 0 less than  $t$  less than 16,700 seconds, which is about 4.5 hours. Now that you've carefully examined this problem using linear approximation, it is probably time to tell your graduate student adviser that you need to order a new batch of  $N_2O_5$  solution. Let's review. A1 The linear approximation to a function at a point  $c$  is the tangent line of a function at  $c$ . A2 This linear approximation only accurately models the function for points sufficiently close to  $c$ . A3 It is easy to read off the derivative at a point from a differential equation, and thus give a linear approximation to the solution. In our example with the concentration of  $N_2O_5$ , we could also solve the differential equation exactly and demonstrate that the approximation was acceptable near the point of approximation  $t=0$ , but not so for larger time intervals. Notice that we used this simple example to illustrate linear approximation because we could also solve it exactly and use the exact solution to determine the error. This is somewhat misleading, since we often want to use linear approximation precisely when the differential equation that describes our physical situation does not have an analytic solution. So you may wonder, is there a way to estimate the error in a linear approximation without knowing the exact solution. The answer is yes! However, it involves Taylor's Remainder theorem, which finds an acceptable range in terms of higher order derivatives, and is beyond the scope this video.