Test 1

18.303 Linear Partial Differential Equations

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1 Rules

You may only use pencils, pens, erasers, and straight edges. No calculators, notes, books or other aides are permitted. Scrap paper will be provided.

Be sure to show a few key intermediate steps when deriving results - answers only will not get full marks.

2 Given

You may assume the eigenvalues of the Sturm-Liouville problem

$$\begin{aligned} X'' + \lambda X &= 0, \qquad 0 < x < 1 \\ X(0) &= 0 \qquad X(1) = 0 \end{aligned}$$

are $\lambda_n = n^2\pi^2$ and $X_n(x) = \sin(n\pi x)$, for $n = 1, 2, ...$, without derivation.

You may also assume the following orthogonality conditions for m, n positive integers:

$$\int_0^1 \sin(m\pi x)\sin(n\pi x)\,dx = \begin{cases} 1/2, & m = n \neq 0, \\ 0, & m \neq n. \end{cases}$$

3 Questions

Total points: 30

Consider the following heat problem in dimensionless variables

$$\begin{aligned} u_t &= u_{xx} - bx, \qquad 0 < x < 1, \qquad t > 0 \\ u(0,t) &= 0, \qquad u(1,t) = 0, \qquad t > 0 \\ u(x,0) &= u_0 \qquad 0 < x < 1, \end{aligned}$$

where $b > 0$ and $u_0 > 0$ are constants. This is the heat equation with a negative source (i.e. extracting heat from the rod).

(a) [3 points] Derive the steady-state (equilibrium) solution

$$u_E(x) = \frac{b}{6} x \left(x^2 - 1\right)$$

It is insufficient to simply verify that the solution works.

(b) [3 points] Using $u_E(x)$, transform the given heat problem for $u(x,t)$ into the following problem for a function $v(x,t)$:

$$\begin{aligned} v_t &= v_{xx}, \qquad 0 < x < 1, \qquad t > 0 \\ v(0,t) &= 0, \qquad v(1,t) = 0, \qquad t > 0 \\ v(x,0) &= f(x) \qquad 0 < x < 1. \end{aligned}$$

where $f(x)$ will be determined by the transformation. Write v_t, v_{xx} in terms of u, b, x. State $f(x)$ in terms of u_0, b and x.

(c) [10 points] Derive the solution

$$v(x,t) = \sum_{n=1}^{\infty} v_n(x,t) = \sum_{n=1}^{\infty} B_n e^{-n^2\pi^2 t} \sin(n\pi x)$$

and derive equations for B_n in terms of $f(x)$. Be sure to give the intermediate steps: separate variables, write down problems and solve for $X(x)$ (using information from the Given section), solve for $T_n(t)$, put things together, impose the IC. Use orthogonality of $\sin(n\pi x)$ (see Given section) to find B_n in terms of $f(x)$. Substitute for $f(x)$ from part (b). You may use (without proof) the fact that

$$\int_0^1 x\left(x^2 - 1\right) \sin(n\pi x)\, dx = \frac{6(-1)^n}{\pi^3 n^3}, \qquad \int_0^1 \sin(n\pi x)\, dx = \frac{1 - (-1)^n}{\pi n}$$

(d) [5 points] Prove that the solution $v(x,t)$ is unique. Recall that $v(x,t)$ satisfies

$$\begin{aligned} v_t &= v_{xx}, \qquad 0 < x < 1, \qquad t > 0 \\ v(0,t) &= 0, \qquad v(1,t) = 0, \qquad t > 0 \\ v(x,0) &= f(x) \qquad 0 < x < 1. \end{aligned}$$

(e) [2 points] Solve for

$$u(x,t) = u_E(x) + \sum_{n=1}^{\infty} u_n(x,t) \tag{1}$$

using the earlier transformations. Write down precisely the functions $u_n(x,t)$.

(f) [4 points] Assume $u_0 = b/\pi^2$ and show that

$$\left| \frac{u_2(x,t)}{u_1(x,t)} \right| \leq \frac{1}{12} e^{-3}, \qquad t \geq 1/\pi^2.$$

(g) [3 points] In (f) you showed that the second term was small compared to the first, so (without proof) write down the first term approximation

$$u(x,t) \approx u_E(x) + A_1 e^{-\pi^2 t} \sin(\pi x)$$

which is expected to be good for $t \geq 1/\pi^2$. Sketch $u = u_0$ and $u = u_E(x)$ for $0 < x < 1$ and comment on the physical significance of the sign of A_1. You may assume $u_0 = b/\pi^2$.