Problem Set 2 : Variations of the Basic Heat Problem

18.303 Linear Partial Differential Equations

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Fall 2004

1 Problem 1

Consider the non-homogeneous heat problem

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}; \qquad u(0,t) = b_0, \qquad u(1,t) = b_1; \qquad u(x,0) = 0$$

where b_0, b_1 are constants.

a. Find the equilibrium solution $u_E(x)$, and transform the problem to a standard homogeneous problem for a temperature function $v(x,t)$.

b. Show that for large t,

$$u(x,t) \approx u_E(x) + Ce^{-\pi^2 t} \sin \pi x$$

Find C.

2 Problem 2

Consider the non-homogeneous heat problem

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + b; \qquad u(0,t) = 0 = u(1,t); \qquad u(x,0) = 0 \tag{1}$$

where $t > 0$, $0 < x < 1$ and b is constant.

a. Find the equilibrium solution $u_E(x)$.

b. Transform the heat problem (1) into a standard homogeneous heat problem for a temperature function $v(x,t)$.

c. Show that after a large time, the solution of the heat problem (1) is approximated by

$$u(x,t) \approx u_E(x) + Ce^{-\pi^2 t}\sin(\pi x).$$

Find C and comment on the physical significance of its sign. Illustrate the solution qualitatively by sketching typical temperature profiles $t =$ constant and the central amplitude profile $x = 1/2$.

3 Problem 3

Transform the heat problem

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}; \qquad u(0,t) = g_1(t); \qquad u(1,t) = g_2(t); \qquad u(x,0) = f(x)$$

with non-homogeneous boundary conditions into a standard problem (i.e. one with homogeneous BCs) in terms of the unknown function $v(x,t)$.

4 Problem 4

Show that if u is a solution of the generalized heat equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + b\frac{\partial u}{\partial x} + cu + g(x,t)$$

where b, c are constants, then

$$v(x,t) = e^{\alpha x + \beta t}u(x,t)$$

satisfies the standard heat equation

$$\frac{\partial v}{\partial t} = \frac{\partial^2 v}{\partial x^2} + h(x,t)$$

for suitable choices of the constants α, β and function $h(x,t)$. In this way, more complicated heat problems can be simplified.

5 Problem 5

Prove that the heat problem

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + h(x,t); \qquad \frac{\partial u}{\partial x}(0,t) = 0 = \frac{\partial u}{\partial x}(1,t); \qquad u(x,0) = f(x)$$

with $t > 0$, $0 \le x \le 1$ has at most one solution (subject to appropriate continuity assumptions).

6 Problem 6

Consider the heat problem with periodic boundary conditions

$$\begin{aligned} u_t &= u_{xx} \\ u(0,t) &= 0; \qquad u(1,t) = \cos \omega t; \qquad t > 0 \\ u(x,0) &= f(x) \qquad 0 < x < 1. \end{aligned}$$

a. Prove that the steady-state solution, $u_{SS}(x,t)$, is unique.

b. Find $u_{SS}(x,t)$ by using the complex change of variable $u_{SS}(x,t) = \operatorname{Re}\{U(x)e^{i\omega t}\}$.

7 Problem 7 Fourier's Ring

Consider a slender homogeneous ring which is insulated laterally. Let x denote the distance along the ring and let l be the circumference of the ring.

a. Show that the temperature $u(x,t)$ satisfies (see Haberman §2.4.2)

$$u_t = \kappa u_{xx}; \qquad u(x+l,t) = u(x,t)$$

b. Introduce a non-dimensional distance and time to the initial value problem

$$\begin{aligned} u_t &= u_{xx}; \qquad 0 < x < 1, \qquad t > 0 \\ u(x+2,t) &= u(x,t); \qquad t > 0 \\ u(x,0) &= f(x) \qquad 0 < x < 1. \end{aligned} \tag{2}$$

Note that your scaling for x will determine the scaled wavelength - find the one that gives you a scaled wavelength of 2.

c. Use separation of variables and Fourier Series to obtain the solution to (2):

$$u(x,t) = A_0 + \sum_{n=1}^{\infty} e^{-n^2\pi^2 t}\left(A_n \cos(n\pi x) + B_n \sin(n\pi x)\right)$$

Give formulae for the coefficients A_n, B_n in terms of $f(x)$.

d. Prove that (2) has at most one solution. Hint: consider $\Delta(t) = \int_0^1 (u_1(x,t) - u_2(x,t))^2\, dx$ where u_1, u_2 are solutions to (2).

8 Problem 8

Consider the two Heat Problems,

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}; \qquad u(0,t) = 0 = u(1,t); \qquad u(x,0) = f(x) \tag{3}$$

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}; \qquad \frac{\partial u}{\partial x}(0,t) = 0 = \frac{\partial u}{\partial x}(1,t); \qquad u(x,0) = f(x) \tag{4}$$

for $t > 0$ and $0 \leq x \leq 1$. Assume $f(x)$ is piecewise smooth on $[0,1]$ and continuous on $(0,1)$.

a. Write down (don't need to derive) the solution for each problem, and list the formulae for the Fourier coefficients.

b. At $t = 0$, you have a Sine Series and a Cosine Series for $f(x)$. Where are these two series equal? Where are they equal to $f(x)$?

c. The point is, you can represent $f(x)$ on $(0,1)$ in multiple ways, but the choice of representations is based on the eigenfunctions that give solutions to the particular Heat Problem.