LECTURE 15

The Vanishing Theorem Implies Cohomological LCFT

Last time, we reformulated our problem as showing that, for an extension L/K of nonarchimedean local fields with Galois group G,

$$(L^\times)^{\mathrm{t}G} \simeq \mathbb{Z}^{\mathrm{t}G}[-2]. \tag{15.1}$$

Thus, our new goal is to compute the Tate cohomology of $L^\times$. Recall that we have let K^{unr} denote the completion of the maximal unramified extension of K; we'd like to use K^{unr} to compute this Tate cohomology.

Claim 15.1. *If $x \in K^{\mathrm{unr}}$ is algebraic over K (which may not be the case due to completion), then $K' := K(x)$ is unramified over K.*

Proof. As a finite algebraic extension of K, K' is a local field, and we have an embedding

$$\mathcal{O}_{K'}/\mathfrak{p}_K\mathcal{O}_{K'} \hookrightarrow \mathcal{O}_{K^{\mathrm{unr}}}/\mathfrak{p}_K\mathcal{O}_{K^{\mathrm{unr}}} = \bar{k},$$

where $k := \mathcal{O}_K/\mathfrak{p}_K$. So $\mathcal{O}_{K'}/\mathfrak{p}_K\mathcal{O}_{K'}$ is a field, hence uniformizers of K and K' are identical. □

Claim 15.2. *$(K^{\mathrm{unr}})^{\sigma=1} = K$, that is, the elements fixed by (i.e., on which it acts as the identity) the Frobenius automorphism $\sigma \in G$ (obtained from the Frobenii of each unramified extension, passed to the completion via continuity).*

Recall that we have a short exact sequence

$$0 \to K \to K^{\mathrm{unr}} \xrightarrow{1-\sigma} K^{\mathrm{unr}},$$

which we may rewrite on multiplicative groups as

$$1 \to K^\times \to K^{\mathrm{unr},\times} \xrightarrow{x \mapsto x/\sigma x} K^{\mathrm{unr},\times} \xrightarrow{v} \mathbb{Z} \to 0.$$

We showed that an element of $K^{\mathrm{unr},\times}$ can only be written as $x/\sigma x$ if it is a unit in the ring of integers $\mathcal{O}_{K^{\mathrm{unr}}}^\times$; this map is an isomorphism on each of the associated graded terms, hence on $\mathcal{O}_{K^{\mathrm{unr}}}^\times$.

Now, we'd like to explicitly construct the isomorphism in (15.1). Our first attempt is as follows: let us assume that L/K is totally ramified (since we discussed the unramified case last time, this is a rather mild assumption), so that $L^{\mathrm{unr}} = L \otimes_K K^{\mathrm{unr}}$. Then we have the following theorem, to be proved later.

Theorem 15.3 (Vanishing Theorem). *If L/K is totally ramified, then the complex $(L^{\mathrm{unr},\times})^{\mathrm{t}G}$ is acyclic.*

Claim 15.4. *The vanishing theorem implies cohomological* LCFT.

Proof. Assume L/K is totally ramified. We have the four-term exact sequence

$$1 \to L^\times \to L^{\mathrm{unr},\times} \xrightarrow{x \mapsto x/\sigma x} L^{\mathrm{unr},\times} \xrightarrow{v} \mathbb{Z} \to 0. \tag{15.2}$$

We may rewrite this as follows:

$$
\begin{array}{ccccccccccc}
A & & \cdots \to & 0 & \longrightarrow & L^\times & \longrightarrow & 0 & \longrightarrow & 0 & \to \cdots \\
\downarrow & & & \| & & \downarrow & & \downarrow & & \| & \\
B & & \cdots \to & 0 & \to & L^{\mathrm{unr},\times} & \xrightarrow{1-\sigma} & L^{\mathrm{unr},\times} & \to & 0 & \to \cdots \\
\downarrow & & & \| & & \downarrow & & \downarrow{\scriptstyle v} & & \| & \\
\mathrm{Coker}(A \to B) & & \cdots \to & 0 & \longrightarrow & 0 & \longrightarrow & \mathbb{Z} & \longrightarrow & 0 & \to \cdots ,
\end{array}
$$

where $L^\times$ is in degree -1. The final quasi-isomorphism to the homotopy cokernel obtained from (15.2) follows from Claim 10.12, because $A \hookrightarrow B$ is an injection (note that this holds in general for any four-term exact sequence). The term-wise cokernel yields an injection

$$L^{\mathrm{unr},\times}/L^\times \xhookrightarrow{x \mapsto x/\sigma x} L^{\mathrm{unr},\times}$$

since, omitting the quotient, $L^\times$ is precisely the kernel of this map.

Now, we have a quasi-isomorphism

$$B^{\mathrm{t}G} = \mathrm{hCoker}\big(L^{\mathrm{unr},\times} \xrightarrow{1-\sigma} L^{\mathrm{unr},\times}\big)^{\mathrm{t}G} \simeq \mathrm{hCoker}\big((L^{\mathrm{unr},\times})^{\mathrm{t}G} \to (L^{\mathrm{unr},\times})^{\mathrm{t}G}\big),$$

so since $(L^{\mathrm{unr},\times})^{\mathrm{t}G}$ is acyclic by the vanishing theorem, this homotopy cokernel is as well by the long exact sequence on cohomology. Thus,

$$(L^\times[2])^{\mathrm{t}G} = \mathrm{hCoker}\big((L^\times[1])^{\mathrm{t}G} \to 0\big) = \mathrm{hCoker}\big(A^{\mathrm{t}G} \to B^{\mathrm{t}G}\big) \simeq \mathbb{Z}^{\mathrm{t}G},$$

as desired. □

Now suppose L/K is a general finite Galois extension with $G := \mathrm{Gal}(L/K)$ (though we could handle the unramified and totally ramified cases separately, as any extension is canonically a composition of such extensions). If L/K is unramified, then

$$L \otimes_K K^{\mathrm{unr}} = \prod_{L \hookrightarrow K^{\mathrm{unr}}} K^{\mathrm{unr}}$$

canonically, indexed by such embeddings. In fact, the following holds:

Theorem 15.5 (General Vanishing Theorem). *$[(L \otimes_K K^{\mathrm{unr}})^\times]^{\mathrm{t}G}$ is acyclic.*

To understand the structure of $L \otimes_K K^{\mathrm{unr}}$, note that we have an action of $\widehat{\mathbb{Z}}\sigma$ on the second factor and of G on the first; these two actions (i.e., $x \otimes y \mapsto gx \otimes y$ and $x \otimes y \mapsto x \otimes \sigma y$) clearly commute. Again, the points fixed under σ are

$$L = L \otimes_K K \hookrightarrow L \otimes_K K^{\mathrm{unr}}.$$

Claim 15.6. *The following sequence is exact:*

$$1 \to L^\times \to (L \otimes_K K^{\mathrm{unr}})^\times \xrightarrow{x \mapsto x/\sigma x} (L \otimes_K K^{\mathrm{unr}})^\times \to \mathbb{Z} \to 0.$$

Proof. If $x \in K^{\mathrm{unr}}$ is a unit, then σx is as well, so the map $x \mapsto x/\sigma x$ is well-defined, and moreover, x is in its kernel if and only if x is fixed under the action of σ, that is, $x \in K$, and since $L \otimes_K K = L$ we obtain a unit of L, which shows exactness of the left half. Now, the map to $\mathbb{Z}$ is defined by

$$\begin{array}{ccc} (L \otimes_K K^{\mathrm{unr}})^\times & \longrightarrow & \mathbb{Z} \\ \quad\searrow^{\mathrm{N}_{L/K}\otimes \mathrm{id}} & & \nearrow^{v}\quad \\ & \underbrace{K \otimes_K K^{\mathrm{unr},\times}}_{K^{\mathrm{unr},\times}}, & \end{array}$$

where

$$\mathrm{N}_{L/K}(x) := \prod_{g \in G} gx.$$

Thus, its kernel is $\mathcal{O}^\times_{K^{\mathrm{unr}}}$, which is precisely the image of $x \mapsto x/\sigma x$. Moreover, the map is surjective as $1 \otimes \pi \mapsto 1$. □

Observe that if L/K is totally ramified, then this is just our extension from before. Indeed, if we write $L^{\mathrm{unr}} = L \otimes_K K^{\mathrm{unr}}$, then the σ's "match up," that is, the induced Frobenius automorphisms of L^{unr} and K^{unr} are identical as L and K have the same residue field. The norm $\mathrm{N}_{L/K} \colon L^{\mathrm{unr},\times} \to K^{\mathrm{unr},\times}$ for this extension satisfies $v_{K^{\mathrm{unr}}} \circ \mathrm{N} = v_{L^{\mathrm{unr}}}$ (such an extension is generated by the nth root of a uniformizer of K, and then $\mathrm{N}(\pi^{1/n}) = \pi$).

Now suppose L/K is unramified of degree n. Fix an embedding $L \hookrightarrow K^{\mathrm{unr}}$, and let $\sigma \in \mathrm{Gal}(L/K)$ also denote the Frobenius element of L/K. Then we have an isomorphism

$$\begin{aligned} L \otimes_K K^{\mathrm{unr}} &\xrightarrow{\sim} \prod_{i=0}^{n-1} K^{\mathrm{unr}} \\ x \otimes y &\mapsto \left((\sigma^i x) \cdot y\right)_{i=0}^{n-1}, \end{aligned}$$

where the product is taken via our fixed embedding (note that this could be done more canonically by taking the product over embeddings as before). We now ask: what does the automorphism $\mathrm{id} \otimes \sigma$ of $L \otimes_K K^{\mathrm{unr}}$ correspond to under this isomorphism? We have

$$x \otimes \sigma y \mapsto (x \cdot \sigma y, \sigma x \cdot \sigma y, \sigma^2 x \cdot \sigma y, \ldots) = \sigma(\sigma^{-1} x \cdot y, x \cdot y, \sigma x \cdot y, \ldots),$$

so it is the action of σ on the rotation to the right of the image of $x \otimes y$ (note that σ doesn't have finite order on K^{unr}, so this should either, which rules our rotation as a possibility for the image of $\mathrm{id} \otimes \sigma$). Similarly, the norm map $\mathrm{N}_{L/K} \colon \prod K^{\mathrm{unr},\times} \to K^{\mathrm{unr},\times}$ takes the product of all entries.

We'd like for some element $(x_0, \ldots, x_{n-1}) \in \prod K^{\mathrm{unr},\times}$ to be in the image of $y/\sigma y$ (i.e., the map in the middle of the exact sequence of Claim 15.6; here σ refers to the automorphism $\mathrm{id} \otimes \sigma$) if and only if $\prod x_i \in \mathcal{O}^\times_{K^{\mathrm{unr}}}$, that is, $\sum v(x_i) = 0$. Recall that the reverse implication is trivial, as we have shown that $\mathcal{O}^\times_{K^{\mathrm{unr}}} \xrightarrow{y/\sigma y} \mathcal{O}^\times_{K^{\mathrm{unr}}}$ is surjective as it is at the associated graded level. For the forward direction, we have

$$(y_0, \ldots, y_{n-1}) \overset{y/\sigma y}{\longmapsto} \left(\frac{y_0}{\sigma y_{n-1}}, \frac{y_1}{\sigma y_0}, \ldots\right) =: (x_0, x_1, \ldots).$$

Thus,

$$y_0 = x_0 \cdot \sigma y_{n-1},$$

$$\begin{aligned} y_1 &= x_1 \cdot \sigma y_0 = x_1 \cdot \sigma x_0 \cdots \sigma^2 y_{n-1}, \\ \cdots &= \cdots \\ y_{n-1} &= x_{n-1} \cdot \sigma x_{n-2} \cdots \sigma^{n-1} x_0 \cdot \sigma^n y_{n-1}, \end{aligned}$$

that is,

$$\frac{y_{n-1}}{\sigma^n y_{n-1}} = x_{n-1} \cdot \sigma x_{n-2} \cdots \sigma^{n-1} x_0.$$

Note that everything here is an element of K^{unr}, so we really do not have $\sigma^n = \text{id}$! Last time, we showed that we can do this if and only if the right-hand side is in $\mathcal{O}_{K^{\text{unr}}}^\times$, which is equivalent to saying that $\sum v(x_i) = 0$. The general case of this exact sequence is sort of a mix of the two.

We now compare these results with those from the last lecture. Assume the Vanishing Theorem. For an unramified extension L/K, we have two quasi-isomorphisms between $(L^\times)^{\text{t}G}$ and $\mathbb{Z}[-2]^{\text{t}G}$, one from what we just did, and the other since $(\mathcal{O}_L^\times)^{\text{t}G} \simeq 0$ implies $(L^\times)^{\text{t}G} \simeq \mathbb{Z}^{\text{t}G} \simeq (\mathbb{Z}[-2])^{\text{t}G}$ by cyclicity. We claim that these two quasi-isomorphisms coincide. A sketch of the proof is as follows: we have $G = \mathbb{Z}/n\mathbb{Z}$ (with generator the Frobenius element), and a short exact sequence

$$0 \to \mathbb{Z} \to \mathbb{Z}[G] \xrightarrow{1-\sigma} \mathbb{Z}[G] \to \mathbb{Z} \to 0.$$

As shown in Problem 1(e) of Problem Set 7, $\mathbb{Z}[G]^{\text{t}G} \simeq 0$ is a quasi-isomorphism (this is easy to show, and we've already shown it for cyclic groups). Thus, we get $\mathbb{Z}^{\text{t}G}[2] \simeq \mathbb{Z}^{\text{t}G}$, and we claim that this is the same isomorphism that we get from 2-periodicity of the complex. The proof is by a diagram chase. We have $(L \otimes_K K^{\text{unr}})^\times = \prod K^{\text{unr},\times}$, which is a finite product. Thus, the diagram

$$\begin{array}{ccccccccccc} 1 & \longrightarrow & L^\times & \longrightarrow & (L \otimes_K K^{\text{unr}})^\times & \xrightarrow{x \mapsto x/\sigma x} & (L \otimes_K K^{\text{unr}})^\times & \xrightarrow{\sum v} & \mathbb{Z} & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle v} & & \downarrow{\scriptstyle \prod v} & & \downarrow{\scriptstyle \prod v} & & \| & & \\ 1 & \longrightarrow & \mathbb{Z} & \longrightarrow & \underbrace{\prod_{i=0}^n \mathbb{Z}}_{\mathbb{Z}[G]} & \xrightarrow{1-\sigma} & \mathbb{Z}[G] & \xrightarrow{\epsilon} & \mathbb{Z} & \longrightarrow & 0 \end{array}$$

commutes, where ϵ denotes the sum over the coordinates of $\mathbb{Z}[G]$. This says precisely that the isomorphisms obtained from both 4-term exact sequences coincide.

The upshot is that under LCFT, we have an isomorphism $K^\times/\mathrm{N}L^\times \simeq \mathbb{Z}/n\mathbb{Z}$ by which $\pi \mapsto \text{Frob}$. Thus, we have reduced LCFT to the Vanishing Theorem, which we will prove in the next lecture.

18.786 Number Theory II: Class Field Theory
Spring 2016