LECTURE 14

Tate Cohomology and K^{unr}

Let G be a finite group and X be a complex of G-modules. Let $P \xrightarrow{\text{qis}} \mathbb{Z}$ be a projective complex of G-modules. Then

- $X^{\mathrm{h}G} := \underline{\mathrm{Hom}}_G(P, X)$ are the *homotopy invariants*;
- $X_{\mathrm{h}G} := P \otimes_{\mathbb{Z}[G]} X$ are the *homotopy coinvariants*;
- $X^{\mathrm{t}G} := \mathrm{hCoker}(X_{\mathrm{h}G} \xrightarrow{\mathrm{N}} X^{\mathrm{h}G})$ is the *Tate complex*.

The former two constructions preserve quasi-isomorphisms, sending acyclic complexes to acyclic complexes, as projective complexes are flat. Moreover, recall the notation

- $H^i(G, X) := H^i(X^{\mathrm{h}G})$ is *group cohomology*;
- $H_i(G, X) := H^{-i}(X_{\mathrm{h}G})$ is *group homology*;
- $\hat{H}^i(G, X) := H^i(X^{\mathrm{t}G})$ is *Tate cohomology*.

The final construction generalizes what we had earlier when G was cyclic if X is in degree 0.

Let us now consider Tate cohomology for modules, and not complexes. Suppose G acts on M. Then giving a map $N \xrightarrow{f} M^G$ for an abelian group N is the same as giving a map $f\colon N \to M$ such that $g \cdot f(x) = f(x)$ for all $g \in G$. Dually, giving a map $M_G \xrightarrow{f} N$ is the same as giving a map $f\colon M \to N$ such that $f(g \cdot x) = f(x)$ (this is because the coinvariants are a quotient of M, whereas the invariants are a submodule). Then since $\mathrm{N}(g \cdot x) = \mathrm{N}(x)$ and $g \cdot \mathrm{N}(x) = \mathrm{N}(x)$ for all $g \in G$, these universal properties yield a diagram

$$\begin{array}{ccc} M & \xrightarrow{\mathrm{N} := \sum g} & M \\ \big\downarrow & & \big\uparrow \\ M_G & \xrightarrow{\mathrm{N}} & M^G, \end{array}$$

where the norm map factors through the invariants and coinvariants. Note that the norm map N is an isomorphism if $\#G$ is invertible in M. Mimicking the definition of Tate cohomology, we get $M^G/\mathrm{N}(M_G) = M^G/\mathrm{N}(M) = \hat{H}^0(G, M)$, so homological algebra is in fact better behaved than our "usual" algebra!

Now we ask: what is the norm map N for a complex of G-modules? We have a canonical composition

$$X_{\mathrm{h}G} = P \otimes_{\mathbb{Z}[G]} X \to \underbrace{\mathbb{Z} \otimes_{\mathbb{Z}[G]} X}_{\substack{\text{term-wise} \\ \text{coinvariants}}} \xrightarrow{\mathrm{N}} \underbrace{\underline{\mathrm{Hom}}_G(\mathbb{Z}, X)}_{\substack{\text{term-wise} \\ \text{invariants}}} \to \underline{\mathrm{Hom}}_G(P, X) = X^{\mathrm{h}G}$$

where the last map is via pullback, and the norm map is applied term-wise via the norm on modules, which we know acts as desired by the previous construction (though it is only defined up to homotopy, etc.). Note that the "term-wise invariants" take the degree-wise "naive" invariants, and don't preserve quasi-isomorphisms; the "term-wise coinvariants" are similar. Altogether, this gives a map which we will call $\mathrm{N}\colon X_{\mathrm{h}G} \to X^{\mathrm{h}G}$.

Taking a complex in degree 0 (and in general, for a complex that is bounded below), the homotopy invariants take that complex further to the right; similarly, coinvariants take that complex leftward. But Tate cohomology does both those things, so the result is unbounded, and tends to be very messy. It can be computed in some simpler cases though, such as the following:

Proposition 14.1. *Let M be a G-module, thought of as a complex in degree 0. Then*

(1) $\hat{H}^i(G, M) = H^i(G, M)$ *if* $i \geq 1$;
(2) $\hat{H}^0(G, M) = M^G/\mathrm{N}(M)$;
(3) $\hat{H}^{-1}(G, M) = \mathrm{Ker}(\mathrm{N})/(g-1) = \mathrm{Ker}(\mathrm{N}\colon M_G \to M)$;
(4) $\hat{H}^{-i}(G, M) = H_{i-1}(G, M)$ *if* $i \geq 2$.

Proof. The composition

$$M_{\mathrm{h}G} \xrightarrow{\mathrm{N}} M^{\mathrm{h}G} \to M^{\mathrm{t}G} = \mathrm{hCoker}(\mathrm{N})$$

yields a long exact sequence on cohomology

$$\cdots \to H_{-i}(M_{\mathrm{h}G}) \to H^i(M^{\mathrm{h}G}) \to \hat{H}^i(G, M) \to H_{-i-1}(M_{\mathrm{h}G}) \to \cdots .$$

If $i \geq 1$, then both $H_{-i}(M_{\mathrm{h}G})$ and $H_{-i-1}(M_{\mathrm{h}G})$ vanish, yielding an isomorphism $H^i(G, M) \simeq \hat{H}^i(G, M)$ by exactness.

Both $H_{-1}(M_{\mathrm{h}G})$ and $H^{-1}(G, M)$ vanish, yielding an exact sequence

$$0 \to \hat{H}^{-1}(G, M) \to M_G \xrightarrow{\mathrm{N}} M^G \to \hat{H}^0(G, M) \to 0,$$

which shows (2) and (3).

If $i \geq 1$, then $H^{-i-1}(G, M)$ and $H^{-i}(G, M)$ vanish, yielding an isomorphism $\hat{H}^{-i-1}(G, M) \simeq H_i(G, M)$ by exactness. □

Thus, cohomology shows up as Tate cohomology in higher degrees, though not in the zeroth degree, and similarly homology shows up except (crucially) in degree 0. Of course, all of this depends on the fact that P is bounded.

Theorem 14.2 (Main Theorem of Cohomological LCFT). *Let L/K be an extension of nonarchimedean local fields with finite Galois group G. Then*

$$(L^\times)^{\mathrm{t}G} \simeq (\mathbb{Z}[-2])^{\mathrm{t}G}.$$

While it's not immediately clear how to construct this isomorphism, it's actually not too complicated! In the next lecture, we'll reduce it to a (very canonical!) vanishing statement.

Now, what does this theorem actually mean? Taking zeroth cohomology, we obtain

$$K^\times/\mathrm{N}(L^\times) = \hat{H}^0(G, L^\times) = H^0((L^\times)^{\mathrm{t}G}) \simeq H^0((\mathbb{Z}[-2])^{\mathrm{t}G}) = \hat{H}^{-1}(G, \mathbb{Z}) = H_1(G, \mathbb{Z}) \simeq G^{\mathrm{ab}},$$

as proven last lecture. We saw that this was true for degree-2 extensions of local fields, so this provides a huge generalization of the Hilbert symbol for local fields!

We now recall the construction of the isomorphism $H_1(G, \mathbb{Z}) \simeq G^{\mathrm{ab}}$. First, we showed that $H_i(G, \mathbb{Z}[G]) = 0$ for all $i > 0$. Indeed,

$$\mathbb{Z}[G] \otimes_{\mathbb{Z}[G]} P \simeq \mathbb{Z}[G] \otimes_{\mathbb{Z}[G]} \mathbb{Z} = \mathbb{Z}$$

is a quasi-isomorphism, since P is quasi-isomorphic to $\mathbb{Z}$. In particular, all of the lower homology groups vanish, since $\mathbb{Z}$ is in degree 0. Then we formed the following short exact sequence:

$$0 \to I_G \to \mathbb{Z}[G] \xrightarrow{\epsilon} \mathbb{Z} \to 0$$

where ϵ is defined by $g \mapsto 1$, and where we have let $I_G := \mathrm{Ker}(\epsilon)$, which is an ideal inside $\mathbb{Z}[G]$ where the sum of all coefficients is zero. Taking the long exact sequence on group homology, we obtain

$$\underbrace{H_1(G, \mathbb{Z}[G])}_{0} \to H_1(G, \mathbb{Z}) \to I_G/I_G^2 \to \mathbb{Z} \xrightarrow{\sim} \mathbb{Z} \to 0,$$

since in general $M_G = M/I_G M$, hence we have an isomorphism $H_1(G, \mathbb{Z}) \simeq I_G/I_G^2$. Finally, we show that $I_G/I_G^2 \simeq G^{\mathrm{ab}}$ by construction maps in both directions. We claim that the following is a homomorphism modulo I_G^2:

$$\begin{array}{ccc} G & \xrightarrow{g \mapsto g-1} & I_G/I_G^2 \\ & \searrow \quad \nearrow & \\ & G^{\mathrm{ab}}. & \end{array}$$

To show this, expand $(g-1)(h-1)$ for $g, h \in G$, and so forth; since I_G/I_G^2 is abelian, this map factors through G^{ab}. For the inverse, we have the composition

$$I_G/I_G^2 \hookrightarrow \mathbb{Z}[G]/I_G^2 \xrightarrow{\sum n_g g \mapsto \prod g^{n_g}} G^{\mathrm{ab}},$$

where we have written G multiplicatively; we can check that this is a homomorphism and an inverse of the previous map.

EXAMPLE 14.3. Suppose L/K is an unramified extension of local fields, so $G := \mathrm{Gal}(L/K) = \mathbb{Z}/n\mathbb{Z}$ is cyclic, where this isomorphism is canonical with the Frobenius element corresponding to 1. Recall that $\hat{H}^0(G, \mathcal{O}_L^\times) = 0$, i.e., $\mathrm{N}\colon \mathcal{O}_L^\times \twoheadrightarrow \mathcal{O}_K^\times$ is surjective. We proved this via filtering; the first subquotient gives the norm map $k_L^\times \xrightarrow{\mathrm{N}} k_K^\times$ and the rest give the trace map $k_L \xrightarrow{\mathrm{T}} k_K$ both of which are surjective (for instance, the latter is because the extension is separable). We also showed that the Herbrand quotient was $\chi(\mathcal{O}_L^\times) = 1$, which implies that $\hat{H}^1(G, \mathcal{O}_L^\times) = 0$ as well, i.e., $(\mathcal{O}_L^\times)^{\mathrm{t}G} \simeq 0$ is a quasi-isomorphism. Form the short exact sequence

$$0 \to \mathcal{O}_L^\times \to L^\times \xrightarrow{v} \mathbb{Z} \to 0,$$

where v denotes the normalized valuation on $L^\times$. Taking Tate cohomology then gives a quasi-isomorphism

$$\mathbb{Z}^{\mathrm{t}G}[-2] \simeq (L^\times)^{\mathrm{t}G} \simeq \mathrm{hCoker}\big(0 \to (L^\times)^{\mathrm{t}G}\big) \simeq \mathrm{hCoker}\big((\mathcal{O}_L^\times)^{\mathrm{t}G} \to (L^\times)^{\mathrm{t}G}\big) \simeq \mathbb{Z}^{\mathrm{t}G},$$

by Theorem 14.2, and since Tate cohomology commutes with cones and preserves quasi-isomorphisms. Thus, $(L^\times)^{\mathrm{t}G}$ is 2-periodic. Note that this is canonical, despite requiring a choice of generator, as we may choose the Frobenius element (by which $x \mapsto x^q$).

We now turn to a discussion of general extensions of local fields.

DEFINITION 14.4. K^{unr} is the (p-adic) completion of the maximal unramified extension of $K \subseteq \overline{K}$.

EXAMPLE 14.5. (1) Let $K := \mathbb{F}_q((t))$. The nth unramified extension of K is $K_n = \mathbb{F}_{q^n}((t))$, so the maximal unramified extension of K is

$$\bigcup_{n \geq 1} \mathbb{F}_{q^n}((t)) \subseteq \overline{\mathbb{F}}_q((t)) = K^{\mathrm{unr}}.$$

(2) If $K := \mathbb{Q}_p$, then $K^{\mathrm{unr}} = W(\overline{\mathbb{F}}_p)[1/p]$, where $W(-)$ denotes the ring of Witt vectors.

The basic structure of K^{unr} is thus a "local field" (not in the sense of local compactness, since the residue field is not finite, but in the sense of being a fraction field of a complete DVR) with residue field $\overline{\mathbb{F}}_q$.

Now, letting π be a uniformizer of K, which will continue to be a uniformizer in each K_n (i.e., the degree-n unramified extension of K), $\mathcal{O}_{K^{\mathrm{unr}}}$ is the π-adic completion of $\bigcup_n \mathcal{O}_{K_n}$. Then we have a short exact sequence

$$0 \to \mathcal{O}_{K^{\mathrm{unr}}}^{\times} \to K^{\mathrm{unr},\times} \xrightarrow{v} \mathbb{Z} \to 0,$$

where v is the normalized valuation on K^{unr} (so that $v(\pi) = 1$). We then define

$$\mathrm{Gal}(K^{\mathrm{unr}}/K) := \mathrm{Aut}_{\mathrm{cts}}(K^{\mathrm{unr}}/K),$$

where the latter is the continuous K-automorphisms of K^{unr}. Letting k denote the residue field of K and σ denote the Frobenius element of $\mathrm{Gal}(\bar{k}/k)$, we have

$$\begin{aligned} \mathrm{Gal}(K^{\mathrm{unr}}/K) \simeq \mathrm{Gal}(\bar{k}/k) &\simeq \widehat{\mathbb{Z}}, \\ \sigma &\mapsto 1. \end{aligned}$$

Since

$$K = \{x \in K^{\mathrm{unr}} : x = \sigma x\},$$

we have a resolution

$$0 \to K \to K^{\mathrm{unr}} \xrightarrow{1-\sigma} K^{\mathrm{unr}},$$

and further, a sequence

$$0 \to K^{\times} \to K^{\mathrm{unr},\times} \xrightarrow{x \mapsto x/\sigma x} K^{\mathrm{unr},\times} \xrightarrow{v} \mathbb{Z} \to 0. \tag{14.1}$$

Note that π cannot be in the image of central map since $v(x) = v(\sigma x)$ for all x. This gives us an expression for $K^{\times}$ in terms of $K^{\mathrm{unr},\times}$, which will be our main tool in coming lectures.

CLAIM 14.6. *The sequence (14.1) is exact.*

PROOF. This is true if and only if every $x \in \mathcal{O}_{K^{\mathrm{unr}}}^{\times}$ can be written as $y/\sigma y$ for some $y \in \mathcal{O}_{K^{\mathrm{unr}}}^{\times}$. This amounts to showing that the map

$$\mathcal{O}_{K^{\mathrm{unr}}}^{\times} \xrightarrow{x \mapsto x/\sigma x} \mathcal{O}_{K^{\mathrm{unr}}}^{\times}$$

is surjective. By completeness of the filtration by the maximal ideal (since K^{unr} is complete by definition), it suffices to prove that this is true at the associated graded level. This gives the maps

$$\overline{\mathbb{F}}_q^{\times} \xrightarrow{x \mapsto x/x^q} \overline{\mathbb{F}}_q^{\times} \quad \text{and} \quad \overline{\mathbb{F}}_q \xrightarrow{x \mapsto (1-q)x} \overline{\mathbb{F}}_q.$$

The first is surjective as we can solve $x^{q-1} = 1/y$ for any $y \in \overline{\mathbb{F}}_q^{\times}$, since $\overline{\mathbb{F}}_q$ is algebraically closed. The latter is invertible, as the map is just the identity. $\square$